# The Number of Stable Points of an Infinite-Range Spin Glass Memory

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*This paper finds a rigorous asymptotic expression for the number of stable points of an infinite-range spin glass with independently identically distributed (i.i.d.) zero-mean gaussian exchange interactions. The result also applies to the number of stable points of a Hopfield Memory (a kind of associative memory) when the memory connections are i.i.d. zero-mean gaussians. The result is that the number of stable points is asymptotic to a constant slightly larger than 1 times 2 to a power slightly larger than n/4, where n is the number of spins in the glass, or the length of the n-tuples to be remembered by the memory. The answer is easily derived using simple asymptotic techniques from an exact expression for the probability that an arbitrary ±1 n-tuple of spins is a fixed point. This expression is obtained from the fact that any distribution of joint zero-mean gaussians of given covariances is specified solely by these covariances. This is a far shorter derivation of the result than those existing.*

## I. Introduction

A spin glass (Ref. 1) can be identified with $n$-tuples of ±1's, called *spins* $\sigma_i$, $1 \leqslant i \leqslant n$. Spins $i$ and $j (i \neq j)$ interact via an *exchange interaction* symmetric matrix $J = (J_{ij})$. Spin $i$ "influences" spin $j$ via $J_{ij}$, and $j$ influences $i$ likewise via $J_{ji} = J_{ij}$; $J_{ii} = 0$, $1 \leqslant i \leqslant n$. This means that, in the absence of other interactions, spin value $\sigma_i$ changes to (or remains equal to) the value

$$\sigma_i' = \mathrm{Sgn}\left(\sum_{j=1}^{n} J_{ij}\sigma_j\right) \tag{1}$$

where the Sgn of a nonzero real number is its sign. We ignore the possibility that the sum in Eq. (1) is 0, for this event will have probability 0 in our model, where the $J_{ij}$, $i \neq j$, are independent identically distributed (i.i.d.), zero-mean gaussian, $n(n-1)/2$ in number. Thus $\sigma_i' = \pm 1$ accordingly as $\Sigma_{j=1}^{n} J_{ij}\sigma_j$ is positive or negative. An $n$-tuple of ±1's,

$$\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$$

is then a fixed point of the spin glass specified by the $n \times n$ symmetric matrix

$$J = (J_{ij})$$

provided

$$\sigma' = \sigma$$

i.e.,

$$\sigma_i' = \sigma_i, 1 \leqslant i \leqslant n \tag{2}$$

We ask, if the $J_{ij}$, $i \neq j$, are zero-mean i.i.d. gaussian (of positive variance), what is the expected number of fixed points in Eq. (2)?

Another formulation is to find the number of stable points of the Hopfield Memory (Ref. 2). In this, $n$-tuples $\sigma$ of $\pm 1$'s are input to an electronic device that connects each input to each other input via a conductance $J_{ij}$, where $J_{ij} = J_{ji}$, and all $J_{ii}$ are zero. The $n$ resulting voltages

$$\sum_j J_{ij} \sigma_j$$

are each hard-limited to become the new $\sigma$ $n$-tuple, but again the changes occur at random rather than at synchronous times. See Ref. 3 for further details on the Hopfield Memory and its potential capacity as an associative or content-addressable memory in processing systems.

From Eqs. (1) and (2), $\sigma$ is a fixed point if and only if

$$\mathrm{Sgn}\left(\sum_j J_{ij}\sigma_j\right) = \sigma_i, 1 \leqslant i \leqslant n \tag{3}$$

This is also the fixed point formula if all components change at once; the difference between the two concepts of fixed points is not relevant in what follows.

## II. Reduction to a Positivity Condition

We claim that the probability that $\sigma$ be a fixed point is the same for all $\sigma$. This is because all the probabilities are equal to the one for $\sigma = (1, 1, \cdots, 1)$. Merely replace $J_{ij}$ by $J_{ij}\,\mathrm{Sgn}(\sigma_i) \cdot \mathrm{Sgn}(\sigma_j)$ so that Eq. (3) can be rewritten

$$\mathrm{Sgn}\left(\left(\sum_j J_{ij}\,\mathrm{Sgn}(\sigma_i)\,\mathrm{Sgn}(\sigma_j)\ \frac{\sigma_j}{\mathrm{Sgn}\sigma_j}\right)\Big/\mathrm{Sgn}(\sigma_i)\right) = \sigma_i \tag{4}$$

In Eq. (4), $\sigma_j/\mathrm{Sgn}\sigma_j = 1$, $1 \leqslant j \leqslant n$, and $\sigma_i\,\mathrm{Sgn}(\sigma_i) = 1$. So Eq. (4) can be written

$$\mathrm{Sgn}\left(\sum_j J_{ij}\,\mathrm{Sgn}(\sigma_i)\,\mathrm{Sgn}(\sigma_j)\right) = 1, 1 \leqslant j \leqslant n \tag{5}$$

This is Eq. (3) with all $\sigma_i = +1$ and $J_{ij}$ replaced by the $n(n-1)/2$ random variables

$$J_{ij}\,\mathrm{Sgn}(\sigma_i)\,\mathrm{Sgn}(\sigma_j)$$

identically distributed with the $J_{ij}$.

The expected number of fixed points $F_n$ is then the probability that the *all* 1's $n$-tuple is fixed, times the number of $n$-tuples $2^n$:

$$F_n = 2^n\, pr\left(\sum_{j=1}^{n} J_{ij} \geqslant 0, 1 \leqslant j \leqslant n\right) \tag{6}$$

Here we recall that the $J_{ij} = J_{ji}$ are $n(n-1)/2$ independent zero-mean gaussians of the same positive variance, which we take as 1; $J_{ii} = 0$, $1 \leqslant i \leqslant n$.

## III. Covariances

What is the Covariance Matrix of two row sums of $J$,

$$X_i = \sum_{j=1}^{n} J_{ij}$$

and

$$X_k = \sum_{j=1}^{n} J_{kj} \quad ?$$

The means are 0, because each $J_{ij}$ has mean 0. The variances ($k = i$) are

$$E\left(X_i^2\right) = n - 1 \tag{7}$$

because there are $n - 1$ independent random variables being added, each of variance 1 (recall $J_{ii} = 0$, all $i$).

What is $\mathrm{Cov}(X_i, X_k)$, $i \neq k$? We claim that any two rows of the symmetric matrix $J$ of exchange interactions has exactly one $J$-random variable in common, so that the rest are independent. If we write the $J_{xy}$ always with $x < y$, row $i$ (with the 0 being in position $i$) is

$$(J_{1i} J_{2i} \cdots J_{i-1,i}\, 0\, J_{i,i+1} \cdots J_{in})$$

while row $k$ is

$$(J_{1k}\, J_{2k} \cdots J_{k-1,k}\, 0\, J_{k,k+1} \cdots J_{kn})$$

We can let $i < k$ without loss of generality. Then the only random variable in common between the two rows is $J_{ik}$, by inspection. So

$$\mathrm{Cov}(X_i, X_k) = E(X_i X_k) = 1, i \neq k \qquad (8)$$

This is because the cross terms

$$E(J_{hi} J_{mk}) = E(J_{hi} J_{kn}) = E(J_{ip} J_{qk}) = E(J_{ip} J_{kr}) = 0$$

in every other case, because they involve mean-0 independent random variables.

## IV. Equivalent Gaussians

Any $n$ zero-mean gaussians of the same variances (Eq. (7)) and covariances (Eq. (8)) will do just as well (Ref. 4, Sec. 9.3) in order to find the probability

$$p_n = pr(X_i > 0, 1 \leqslant i \leqslant n)$$

This, from Eq. (6), is $2^{-n}$ times the desired answer $F_n$ for the expected number of fixed points of the random gaussian spin glass:

$$F_n = 2^n\, pr\,(X_i > 0, 1 \leqslant i \leqslant n) = 2^n\, p_n \qquad (9)$$

Here is another way of getting zero-mean gaussians of the same covariances (Eqs. (7) and (8)). Let $Y_0, Y_1, \cdots, Y_n$ be $n+1$ independent mean-0 variance-1 gaussians. Let

$$S_i = \sqrt{n-2}\, Y_i - Y_0, 1 \leqslant i \leqslant n \qquad (10)$$

These are, of course, zero-mean joint gaussians (for $n > 2$). What are their covariances?

We have

$$E(S_i^2) = (n-2)\, E(Y_i^2) + 1$$

the cross-terms $\sqrt{n-2}\, E(Y_i Y_0)$ vanishing. Thus

$$E(S_i^2) = n - 1, 1 \leqslant i \leqslant n \qquad (11)$$

Likewise,

$$E(S_i S_k) = 1, 1 \leqslant i, k \leqslant n, i \neq k \qquad (12)$$

From Eqs. (7) and (8), the $n$ gaussian random variables $S_i$ have the same covariance matrix as the $n$ guassian random variables $X_i$, and so we may use them instead of the $X_i$ to calculate $F_n$ in Eq. (9). This we do in the next section.

## V. Exact Answer

From Eq. (9), we want

$$F_n = 2^n\, pr(S_i > 0, 1 \leqslant i \leqslant n) = 2^n\, p_n \qquad (13)$$

From Eq. (10), this can be written (for $n > 2$) as

$$F_n = 2^n\, pr\,(Y_i > Y_0/\sqrt{n-2}, 1 \leqslant i \leqslant n) \qquad (14)$$

Since the $n+1$ $Y_i$'s are jointly independent, the $n$ random variables $Y_1, Y_2, \cdots, Y_n$ are *conditionally* independent given that $Y_0 = y$. So

$$pr(Y_1 > Y_0/\sqrt{n-2}, 1 \leqslant i \leqslant n \,|\, Y_0 = y) = [Q(y/\sqrt{n-2})]^n \qquad (15)$$

by the conditional independence. Here $Q$ is the righthand tail of the standard gaussian:

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_{t=z}^{\infty} e^{-t^2/2}\, dt \qquad (16)$$

We can average Eq. (15) over $y$, which has the standard gaussian distribution, to get the unconditional probability that $Y_i \geqslant Y_0/\sqrt{n-2}, 1 \leqslant i \leqslant n$. The result (for $n > 2$) is

$$p_n = pr(Y_i \geqslant Y_0/\sqrt{n-2}, 1 \leqslant i \leqslant n)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} Q^n\left(\frac{t}{\sqrt{n-2}}\right) e^{-t^2/2}\, dt$$

$$= -\int_{t=-\infty}^{\infty} Q^n\left(\frac{t}{\sqrt{n-2}}\right) dQ(t) \tag{17}$$

This is exact. For $n = 2, p_n = 1/2 = pr(J_{12} > 0, J_{21} > 0)$. (This follows without this method.) We conclude that for $n = 2$, there are $F_2 = 2$ fixed points on the average. For $n = 3$, we use $n - 2 = 1$ in Eq. (17) to find

$$p_3 = -\int_{t=-\infty}^{\infty} Q^3(t)\, dQ(t) = -\frac{Q^4(t)}{4}\Bigg|_{t=-\infty}^{\infty} = \frac{1}{4}$$

Thus $F_3 = 8p_3 = 2$ also.

## VI. Asymptotic Form

To find $F_n$ for large $n$, let

$$s = t/\sqrt{n-2}$$

in Eq. (17). The result is

$$p_n = \sqrt{\frac{n-2}{2\pi}} \int_{s=-\infty}^{\infty} Q^n(s)\, e^{-(n-2)s^2/2}\, ds$$

or

$$p_n = \sqrt{\frac{n-2}{2\pi}} \int_{s=-\infty}^{\infty} \left[Q(s)\, e^{-s^2/2}\right]^n e^{s^2}\, ds \tag{18}$$

This is now in a form ripe for the Laplace or saddle point method (Ref. 5, Sec. 4.2). The method implies that for suitable functions $g$ and $h$ (the conditions are satisfied here with $e^{h(s)} = Q(s)\, e^{-s^2/2}$, $g(s) = e^{s^2}$), we have the asymptotic equality

$$\int_{s=-\infty}^{\infty} e^{nh(s)} g(s)\, ds \sim \frac{e^{nh(s_0)}}{\sqrt{n}} \left\{ g(s_0) \sqrt{\frac{2\pi}{-h''(s_0)}} \right\} \tag{19}$$

In this,

$$s_0 = \arg\max h(s) = \arg\max e^{h(s)} \tag{20}$$

is to be the unique maximum, and we must have

$$h''(s_0) < 0 \tag{21}$$

conditions which we will check shortly.

Assuming all this, we have from Eqs. (18) and (19) for $n$ large

$$p_n \sim \left(Q(s_0)\, e^{-s_0^2/2}\right)^n e^{s_0^2} \Big/ \left(\frac{d^2}{ds^2} \left(\log Q(s)\, e^{-s^2/2}\right)\right)_{s=s_0} \tag{22}$$

The next section shows

$$s_0 = -0.50605$$

$$Q(s_0)\, e^{-s_0^2/2} = 0.61023$$

$$e^{s_0^2} \Big/ \left(\frac{d^2}{ds^2} \left(\log Q(s)\, e^{-s^2/2}\right)\right)_{s=s_0} = 1.0505$$

and so

$$p_n \sim (1.0505)(0.6102)^n \tag{23}$$

From Eq. (13), then,

$$F_n = 2^n p_n \sim (1.0505)(1.22046)^n$$

or, in more familiar form,

$$F_n \sim (1.0505) 2^{0.2874n} \tag{24}$$

the desired result.

We note here that the exponent in Eq. (24) agrees with that in Ref. 1. But Ref. 1 had at best a logarithmically asymptotic answer, not a true asymptotic one, due both to minor errors and to the method there. However, Ref. 6 has the correct result, although the constants are not specifically worked out and the proof is much more complicated (although more generalizable) than the one given here.

## VII. Calculation Details

This last section gives some details of the calculations of the preceding section and presents numerical results based on evaluating Eq. (18). Note that there is a finite maximum for $h$ because $Q(-\infty) = 1, Q(\infty) = 0$, and $h(\pm\infty) = -\infty$. We have

$$h(s) = \log Q(s) - s^2/2, g(s) = s^2 \tag{25}$$

Thus,

$$h'(s) = -\frac{Z(s)}{Q(s)} - s \tag{26}$$

where

$$Z(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \tag{27}$$

is the unit normal density function. From Eq. (26),

$$h''(s) = \frac{sZ(s)}{Q(s)} - \left(\frac{z(s)}{Q(s)}\right)^2 - 1 \tag{28}$$

To search for a maximum of $h(s)$, we set $h'(s)$ equal to 0 in Eq. (26) to obtain

$$\frac{Z(s_0)}{Q(s_0)} + s_0 = 0 \tag{29}$$

as the condition on $s_0$ for the derivative $h'(s_0)$ to be zero. We see then that $s_0 < 0$. Continuing, from Eq. (28),

$$h''(s_0) = -2s_0^2 - 1 < 0 \tag{30}$$

Thus, there is a *unique* maximum on $(-\infty, \infty)$ and the conditions for Eq. (19) to hold are satisfied.

From Eqs. (18), (19), (29), (30), and (9), we now know that

$$F_n \sim \left(\sqrt{\frac{2}{\pi}} e^{-s_0^2}/|s_0|\right)^n e^{s_0^2} / \sqrt{1 + 2s_0^2} \tag{31}$$

So, what is $s_0$? Calculations on a good personal computer show

$$\begin{aligned} s_0 &= -0.50605 \\ \sqrt{\frac{2}{\pi}} e^{-s_0^2}/|s_0| &= 1.22046 \\ e^{s_0^2} / \sqrt{1 + 2s_0^2} &= 1.0505 \end{aligned} \tag{32}$$

This completes the derivation of Eq. (24).

Table 1 gives comparisons between a numerical evaluation of exact expression Eq. (17) times $2^n$ with the asymptotic formula Eq. (24). For $n = 2$ and 3, the exact answers, as we saw in Sec. V, are both 2. Note how good the asymptotic values for the expected number of fixed points are even for small $n$.

# References


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**Table 1. Comparison of asymptotic and exact values of $F_n$**

| | $F_n$ | |
|---|---|---|
| $n$ | Exact | Asymptotic |
| 2 | 2 | 1.56 |
| 3 | 2 | 1.91 |
| 4 | 2.40 | 2.33 |
| 5 | 2.90 | 2.84 |
| 6 | 3.53 | 3.47 |
| 7 | 4.29 | 4.24 |
| 8 | 5.23 | 5.17 |
| 9 | 6.37 | 6.31 |
| 10 | 7.76 | 7.70 |
| 20 | 56.69 | 56.48 |
| 100 | 4.72E8 | 4.72E8 |