The Effects of Correlated Noise in Intra-Complex DSN Arrays for S-Band Galileo Telemetry Reception

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A number of the proposals for supporting a Galileo S-band (2.3-GHz) mission involve arraying several antennas to maximize the signal-to-noise ratio (and bit rate) obtainable from a given set of antennas. Arraying is no longer a new idea, having been used successfully during the Voyager encounters with Uranus and Neptune. However, arraying for Galileo’s tour of Jupiter is complicated by Jupiter’s strong radio emission, which produces correlated noise effects. This article discusses the general problem of correlated noise due to a planet, or other radio source, and applies the results to the specific case of an array of antennas at the DSN’s Tidbinbilla, Australia, complex (DSS 42, DSS 43, DSS 45, and the yet-to-be built DSS 34). The effects of correlated noise are highly dependent on the specific geometry of the array and on the spacecraft–planet configuration; in some cases, correlated noise effects produce an enhancement, rather than a degradation, of the signal-to-noise ratio. For the case considered here—an array of the DSN’s Australian antennas observing Galileo and Jupiter—there are three regimes of interest. If the spacecraft–planet separation is ≤ 75 arcsec, the average effect of correlated noise is a loss of signal to noise (~0.2 dB as the spacecraft–planet separation approaches zero). For spacecraft–planet separations ≥ 75 arcsec, but ≤ 400 arcsec, the effects of correlated noise cause signal-to-noise variations as large as several tenths of a decibel over time scales of hours or changes in spacecraft–planet separation of tens of arcseconds; however, on average its effects are small (<0.01 dB). When the spacecraft is more than 400 arcsec from Jupiter (as is the case for about half of Galileo’s tour), correlated noise is a <0.05-dB effect.

I. Introduction

It is now becoming common practice to array a number of antennas for telemetry reception in order to increase the effective aperture [1,2]. The signal-to-noise ratio (SNR) improvement obtained by arraying is straightforward to calculate when a pointlike spacecraft is the only source in the array’s field of view. However, the calculation becomes complicated when an additional hot body (such as a radio-loud planet) is in the beam since, unlike system noise, the incoming radiation from this hot body is correlated at different antennas. This article presents a calculation
of the effect of correlated noise on the SNR achieved by a
given array configuration. Other aspects of this topic have
been discussed in [3] and in references therein.

This article is organized as follows: Sections II and III
present a general formulation of the problem, Section IV
considers the special case of an array of $N$ identical anten-
as, and Section V looks at the specific case of arraying the
dSN complex in Australia for Galileo S-band (2.3-GHz)
telemetry reception. To make the article more readable,
many of the detailed calculations have been relegated to
the appendices. Much of the discussion parallels Hjellming's
treatment of the Very Large Array (VLA) [4].

II. Outline of the Problem

Consider an array of $N$ (not necessarily identical) anten-
as. The on-axis gain* of the $i$th antenna is denoted by
$G_i$ and its system temperature (looking at an empty field)
is denoted by $T_i$. The direction toward a point on the sky
can be described by a unit vector $\hat{s}$, with $\hat{s}_0$ pointing
toward a source at the array's field center. The shape of the
$i$th antenna beam can be described by the field pattern, a
dimensionless function $f_i(\hat{s} - \hat{s}_0)$ such that $f_i^2(\hat{s} - \hat{s}_0)G_i$
describes the effective gain of the $i$th antenna in a direc-
tion $\hat{s} - \hat{s}_0$ from the beam center. In general, $G_i$, $T_i$, and
$f_i$ may all be elevation dependent. This dependence is not
considered here; it can easily be included in quantitative
calculations. The geometry of the array of $N$ antennas
can be described by the set of baseline vectors between
each antenna pair $\mathbf{B}_{ik} = -\mathbf{B}_{ki}$, where $\mathbf{B}_{ik}$ denotes the
vector from the $i$th antenna to the $k$th antenna. The basic
geometry is shown in Fig. 1.

It is assumed that the spacecraft is transmitting a narrow-band, polarized signal with a total Earth-received
power per unit area of $P_s/c$ and that the planet has a to-
tal, unpolarized flux density (power per unit area per unit
bandwidth) $S_P$. The angular distribution of the planet's
radio emission is described by the brightness distribution

(flux density per unit solid angle) $I_\nu(\hat{s})$, which is assumed
to be constant in time and only slowly varying with fre-
cquency (i.e., constant across the observing bandwidth).

Fairly simple assumptions are made about the method
of arraying: The incoming RF signal at the $i$th antenna is
mixed to baseband, after which a phase shift is inserted.
This phase shift has a component $\phi_i^m$ (the model phase)
that can be used to compensate, in real time, for phase
differences between antennas (see Appendices A and C for
details). The baseband signal is delayed by a model delay
$\tau_i^m$, and finally, the delayed signals from the $N$ antennas
are summed with appropriate weighting (see Appendix D).
It is assumed that the signal is filtered through a rectan-
gular baseband filter of width $\Delta \nu_i$ for an upper sideband
system (which is assumed throughout), the sky frequency
at the center of the passband is $\nu_{s,ky} = \nu_0 + \Delta \nu_i/2$, where
$\nu_0$ is the mixer frequency. It is also assumed that the
model delays are chosen so as to compensate for the delay
that a signal from the direction $\hat{s}_0$ (the field center) suffers
between an arbitrary reference point and the $i$th antenna.

$$
\tau_i^m = \frac{\mathbf{r}_i \cdot \hat{s}_0}{c}
$$

where $\mathbf{r}_i$ is the vector from the reference point to antenna
$i$. A simple block diagram of the signal path is shown in
Fig. 2.

The total arrayed power can be written as the sum of
the arrayed power from each of the three sources $P_\Sigma =
P_{s,c}^l + P_{s,c}^b + P_N^s$, where $P_{s,c}^l$, $P_{s,c}^b$, $P_N^s$ represent, re-
tively, the arrayed power from the spacecraft, from the
planet, and from system noise. The effective SNR of the
array is proportional to the ratio of the arrayed space-
craft power to the total arrayed power; in the low signal
limit, this reduces to the ratio of $P_{s,c}^l$ to the power re-
ceived from all sources other than the spacecraft (in this
case, the planet and system noise). A figure of merit $\beta$
can be defined for an array; $\beta$ is analogous to $G/T$ for a
single dish and quantifies how, for a given spacecraft power
($P_s/c$) an observing bandwidth ($\Delta \nu$), the SNR varies with
the following array configuration:

$$
\beta = \frac{\Delta \nu}{2P_s/c} \frac{P_{s,c}^l}{P_\Sigma} \approx \frac{\Delta \nu}{2P_s/c} \frac{P_{s,c}^l}{P_s^l + P_N^s}
$$

1 Throughout this article, an astronomer's definition of antenna gain
is used (commonly measured in units of K per Jy) of $G = \epsilon A/2k_B$
where $\epsilon$ is the dimensionless antenna efficiency, $A$ is the physical
antenna area, and $k_B$ is Boltzmann's constant.

2 Here, $\hat{s}_0$ is defined as the nominal direction in which the antennas
in the array are pointing, which is the direction assumed when
model delays are calculated (see Appendix A). As discussed in
Appendix C, it is not necessarily the phase center of the array.

3 In general, the field pattern is considered to be a complex quantity
(e.g., [5, p. 27]), but for the purposes of this article, it is sufficient
to assume that it is real.

4 System noise includes receiver noise, sky background, and ground
pickup—all sources of unwanted noise other than individual, iden-
tifiable sources (such as a planet) in the field of view.
The term \( P_{\Sigma}^{c}/P_{\Sigma}^c \) is the SNR; the term \( \Delta \nu / 2 \) \( P_{\Sigma}^{c} \) is a normalization factor that removes from \( \beta \) any dependence on quantities other than array and source geometry.

### III. General Expression for \( \beta \)

As discussed in Appendix A, the summed baseband voltage can be written in the form

\[
v_{\Sigma}(\nu, t) = \sum_{i=1}^{N} W_i e^{-i\Delta \nu i} v_i(\nu, t + \tau_i^m)
\]

where \( v_i(\nu, t) \) is the baseband voltage given by Eq. (A-4), \( \nu \) is the baseband frequency, and \( W_i \) is the weighting factor.

The arrayed power is then

\[
P_{\Sigma} = \left\langle \int_{0}^{\Delta \nu} \int_{0}^{\Delta \nu} \left| v_{\Sigma}(\nu, t) \right|^2 \right\rangle_T
\]

\[
= \left\langle \int_{0}^{\Delta \nu} \int_{0}^{\Delta \nu} \left| \sum_{i=1}^{N} W_i e^{-i\Delta \nu i} v_i(\nu, t + \tau_i^m) \right|^2 \right\rangle_T
\]

\[
= \sum_{i=1}^{N} W_i^2 P_i + \sum_{i=1}^{N} \sum_{k \neq i} W_i W_k \rho_{ik}
\]

where the angle brackets denote time averaging and \( T \) is the interval over which the time average is taken; \( P_i \) is the single dish power from the \( i \)th antenna,

\[
P_i = \left\langle \int_{0}^{\Delta \nu} \int_{0}^{\Delta \nu} \left| v_i(\nu, t) \right|^2 \right\rangle_T
\]

and \( \rho_{ik} \) is the unnormalized correlation between the \( i \)th and \( j \)th antennas,

\[
\rho_{ik} = \left\langle \int_{0}^{\Delta \nu} \int_{0}^{\Delta \nu} e^{i(\Delta \nu i - \Delta \nu j)} v_i(\nu, t + \tau_i^m) v_k(\nu, t + \tau_k^m) \right\rangle_T
\]

As can be seen in Eqs. (A-11) and (A-18), both \( P_i \) and \( \rho_{ik} \) can be written as the sum of contributions from the spacecraft (\( P_{\Sigma}^{c}, \rho_{ik}^{c} \)), the planet (\( P_{P}^c, \rho_{P}^c \)), and random noise (\( P_{N}^c, \rho_{N}^c \)), so Eq. (4) can be written

\[
P_{\Sigma} = \sum_{i=1}^{N} W_i^2 (P_i^{c} + P_{P}^c + P_{N}^c)
\]

\[
+ \sum_{i=1}^{N} \sum_{j \neq i} W_i W_k (\rho_{ik}^{c} + \rho_{ik}^{P} + \rho_{ik}^{N})
\]

Thus, \( P_{\Sigma} \), the total power in the arrayed signal, can be written as the sum of contributions from the three sources (the spacecraft, the planet, and system noise), \( P_{\Sigma} = P_{\Sigma}^{c} + P_{P}^c + P_{N}^c \), with

\[
P_{\Sigma}^{c} = \sum_{i=1}^{N} W_i^2 P_i^{c} + \sum_{i=1}^{N} \sum_{k \neq i} W_i W_k \rho_{ik}^{c}
\]

\[
P_{P}^c = \sum_{i=1}^{N} W_i^2 P_{P}^c + \sum_{i=1}^{N} \sum_{k \neq i} W_i W_k \rho_{ik}^{P}
\]

\[
P_{N}^c = \sum_{i=1}^{N} W_i^2 P_{N}^c + \sum_{i=1}^{N} \sum_{k \neq i} W_i W_k \rho_{ik}^{N}
\]

Because the phases of the individual \( \rho_{ik}^{c} \) terms in Eq. (10) are random, their sum is always small (on average, zero) and was dropped. However, the sums over \( \rho_{ik}^{c} \) and \( \rho_{ik}^{P} \) are not necessarily small; in fact, a key aspect of arraying is choosing the inserted model phases to ensure that the sum over \( \rho_{ik}^{P} \) is maximized (see Appendix C). Expressions for \( P_{\Sigma}^{c}, P_{P}^c, \) and \( P_{N}^c \) are derived in Appendix A, Section II, and given by Eqs. (A-13), (A-14), and (A-15), respectively. Expressions for \( \rho_{ik}^{c} \) and \( \rho_{ik}^{P} \) are derived in Appendix A, Section C, and given by Eqs. (A-25) and (A-28). Substituting Eqs. (8)–(10) into Eq. (2) gives

\[
\beta = \frac{\Delta \nu}{2 P_{\Sigma}^{c}} \times \frac{\sum_{i=1}^{N} W_i^2 P_i^{c} + \sum_{i=1}^{N} \sum_{k \neq i} W_i W_k \rho_{ik}^{c}}{\sum_{i=1}^{N} W_i^2 (P_i^{c} + P_{P}^c) + \sum_{i=1}^{N} \sum_{k \neq i} W_i W_k \rho_{ik}^{P}}
\]
The phases of $\rho_{ik}^{\phi/c}$ and $\rho_{ik}^P$ depend on the quantity $\delta \phi = \phi_i - \phi_k - \phi - \phi_k$, where $\phi_i, \phi_k$ are the actual antenna-based phases, which include both hardware and atmospheric effects, and $\phi_i^m, \phi_k^m$ are the model phases. If no attempt is made to compensate for the individual antenna phases (i.e., if $\phi_i^m = \phi_k^m = 0$), the phases of $\rho_{ik}^{\phi/c}$ and $\rho_{ik}^P$ are uncorrelated from baseline to baseline, and in both Eqs. (8) and (9) the sums over the cross-correlation will be small (zero on average). However, as discussed in Appendix C, the total phase of $\rho_{ik}^{\phi/c}$ will be zero for all $i, k$, if the values of $\phi_i^m, \phi_k^m$ are chosen to satisfy Eq. (C-1):

$$
\delta \phi = \phi_i - \phi_k - \phi = \phi_k^m - \phi_i^m - \phi_k + \phi_i
$$

$$
= -2\pi \nu/c \frac{B_{ik}}{c} [s_i - s_k]
$$

When $\phi_i^m, \phi_k^m$ are chosen to satisfy the above expression, all the terms in the double sum in Eq. (8) add in phase, and for large $N$, the sum over $\rho_{ik}^{\phi/c}$ contributes significantly more to $P_{\Sigma}^{\phi/c}$ than does the sum over $P_{\Sigma}^{\phi/c}$ (since $\rho_{ik} \sim \sqrt{P_i P_k}$). Phases chosen to satisfy Eq. (C-1) maximize $P_{\Sigma}^{\phi/c}$, and except in pathological cases this choice maximizes $\beta$. The process of determining and inserting these values of $\phi_i^m, \phi_k^m$ is referred to as phasing the array. In the following, the subscript $\phi$ is used to denote expressions that assume that the array is phased on the spacecraft.

Unfortunately, the values of $\phi_i^m, \phi_k^m$, which maximize the sum over $\rho_{ik}^{\phi/c}$, may also de-randomize the phases of $\rho_{ik}^P$, as a consequence, sums over $\rho_{ik}^P$ in Eqs. (4) and (11) can become significantly nonzero. This is the source of the correlated noise contribution from the planet.

Equations (A-13), (A-14), (A-15), (A-25), and (A-28) can be used to expand Eqs. (8), (9), and (10), and if the array is phased on the spacecraft [Eq. (C-1) is satisfied], the arrayed power from the spacecraft, the planet, and system noise can be written, respectively, as

$$
P_{\Sigma}^{\phi/c} = 2k_B P_{s/c} \left[ \sum_{i=1}^{N} W_i^2 G_i + \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k} \right]
$$

$$
P_{\Sigma}^P = \Delta \nu k_B S_P \left[ \sum_{i=1}^{N} W_i^2 \bar{f}_{iP} \bar{f}_{kP} G_i + \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \bar{f}_{iP} \bar{f}_{kP} \sqrt{G_i G_k} \mathcal{F}_{ik}^P \right]
$$

$$
P_{\Sigma}^N = \Delta \nu k_B \sum_{i=1}^{N} W_i^2 T_i
$$

In the above expressions, $\bar{f}_{iP}, \bar{f}_{kP}$ represent the average value of the beam pattern in the direction of the planet, and $\mathcal{F}_{ik}^P$ [defined by Eq. (A-27)] is a dimensionless, complex quantity, with magnitude less than or equal to unity, which depends only on the array geometry and the structure of the planet (or other background source).

By inserting Eqs. (12), (13), and (14) into Eq. (2), the expression for $\beta$ reduces to

$$
\beta = \frac{\left[ \sum_{i=1}^{N} W_i^2 G_i + \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k} \right]}{S_P \left[ \sum_{i=1}^{N} W_i^2 \bar{f}_{iP} \bar{f}_{kP} G_i + \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \bar{f}_{iP} \bar{f}_{kP} \sqrt{G_i G_k} \mathcal{F}_{ik}^P \right] + \sum_{i=1}^{N} W_i^2 T_i}
$$

IV. An Array of Identical Antennas

The major effects of correlated noise are easily seen by examining an array of identical antennas. Consider an array of $N$ antennas, each with a gain $G$, a system temperature $T$, and an average field pattern in the direction of the planet $\bar{f}_P$. Since the antennas are identical, $W_i = 1$ for all

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On page 132, the note states: One can imagine cases where the increase in $P_{\Sigma}^{\phi/c}$ obtained by phasing the array is more than outweighed by the resulting increase in $P_{\Sigma}^P$, but even attempting to array in such cases would be difficult.
When this array is properly phased on the spacecraft, Eqs. (12), (13), and (14) become

\[ P_{E \phi}^{\beta} = 2k_B N^2 G P_{s/c} \]  
\[ P_{E \phi}^P = \Delta \nu k_B f_p^2 G S_P \left[ N + \sum_{i=1}^{N} \sum_{k \neq i} F_{ik}^P \right] \]  
\[ P_{E \phi}^N = \Delta \nu k_B N T \]

and Eq. (15) becomes

\[ \beta_{\phi} = \frac{NG}{f_p^2 G S_P \left[ 1 + N^{-1} \sum_{i=1}^{N} \sum_{k \neq i} F_{ik}^P \right] + T} \]  

Since \(|F_{ik}^P| \leq 1\) for all baselines, it is always true that

\[ 1 + N^{-1} \sum_{i=1}^{N} \sum_{k \neq i} F_{ik}^P \leq N \]  

From Eq. (A-27) [see also Eqs. (B-1), (B-3), and (B-5)], it can be seen that, in the limit of an extremely compact array [i.e., \(B_{ik} \ll c/(\nu_s^2 \gamma_R P)\), for all \(i, k\); \(R_P\) being the characteristic angular radius of the planet], \(F_{ik}^P \to 1\) on all baselines and

\[ \beta_{\phi_0} \to \frac{NG}{NG S_P + T} \]  

where the substitution \(f_p = 1\), appropriate in this limit,\(^6\) has been made. This is equivalent to observing the spacecraft with a single dish of gain \(NG\) and an effective system temperature of \(NG S_P + T\). Not surprisingly, observing the spacecraft and planet with a compact array is analogous to observing the two objects with an antenna with \(N\) times the area of a single array element.

In the limit where \(B_{ik} > c/(\nu_s^2 \gamma_R P)\), for all \(i, k\) (i.e., an extremely extended array), \(|F_{ik}^P| \to 0\) on all baselines and

\[ \beta_{\phi_0} \to \frac{NG}{f_p^2 G S_P + T} \]  

This is equivalent to observing the spacecraft with a single dish of gain \(NG\) and an effective system temperature of \(f_p^2 G S_P + T\).

In both cases, the effective gain is \(NG\); it is independent of baseline length and depends only on the gains of the individual antennas. However, the effective system temperature is not the same in the two extreme configurations. The planet contributes \(Nf_p^2 G S_P\) to the system temperature in the compact array limit but only \(f_p^2 G S_P\) in the extended array limit. In the extended array limit, the noise contribution from the planet is simply the sum of its (uncorrelated) contributions to the individual system temperature. In the compact array limit, its contribution is \(N\) times larger due to the correlated noise effects.

Though, as discussed below, the effects of correlated noise are a complicated function of array geometry. In general, the more extended the array configuration, the smaller the correlated noise contribution of a planet or other hot body. As a rule of thumb, correlated-noise contributions are significant on baselines where \(\nu_s^2 \gamma_R B_{ik} R_P/c \ll 1\) (see Appendix A, Section III).

It should be noted that in the intermediate cases where \(B_{ik} \sim C/(\nu_s^2 \gamma_R P)\), the sum over \(F_{ik}^P\) may, for some geometries, be negative. In such cases, the performance of the array is actually enhanced over that of the extended array limit.

**V. DSN Complexes at S-Band**

One of the proposals for support of the Galileo S-band mission is the arraying of antennas at each DSN complex. This section describes the performance of an array at the Canberra complex that includes DSS 42, 43, 45, and the soon-to-be-built DSS 34. Table 1 lists each antenna, its S-band gain and system temperature at zenith, its S-band beamwidth at full-width half-power (FWHP), and its station coordinates (east, north, and vertical) relative to DSS 43.

With the model of Jupiter given in Appendix B, Eq. (15) can be used to calculate \(\beta_{\phi}\) for the array. The improvement that the array would provide relative to the stand-alone use of DSS 43, if correlated noise effects could be neglected, is given by the ratio

---

\(^6\)If the baseline is small compared to \(c/(\nu_s^2 \gamma_R P)\), the diameters of the individual antennas would be smaller still.
\[
\beta_{\phi_0} / \beta_{43} = \left( \frac{\sum_{i=1}^{N} W_i^2 G_i + \sum_{i=1}^{N} \sum_{j \neq i} W_i W_j \sqrt{G_i G_j}}{S_P \sum_{i=1}^{N} W_i^2 f_{iP}^2 G_i + \sum_{i=1}^{N} W_i^2 T_i} \right) \times \frac{T_{43} + \bar{f}_{43}^2 G_{43} S_P}{G_{43}}
\] (23)

When correlated noise is neglected, this ratio is independent of hour angle, array geometry, or source structure, but it depends on \( S_P \)—the flux of Jupiter—and, therefore, on the Earth–Jupiter distance. At opposition, this distance is 4.2 AU, the S-band flux of Jupiter is approximately 5.8 Jy and, with the parameters given in Table 1, \( \beta_{\phi_0} / \beta_{43} = 1.41 \). At conjunction (an Earth–Jupiter distance of 6.2 AU), \( S_P \approx 2.6 \) Jy and \( \beta_{\phi_0} / \beta_{43} = 1.38 \). The improvement is not as great at large Earth–Jupiter distances because Jupiter’s contribution to the system temperature is less; for comparison, if \( S_P = 0 \), \( \beta_{\phi_0} / \beta_{43} = 1.34 \). It is useful to note that despite the effects of correlated noise, averaging is most useful when extended background sources are present, particularly if the baselines are long. In the above calculations, it is assumed that \( f_P = 1 \) for all antennas and, as in all the calculations in this section, \( W_i = \sqrt{G_i} / T_i \) [see Eq. (D-4)]. Throughout this section, it is also assumed that the gains \( G_i \) and system temperatures \( T_i \) do not vary with elevation, which leads to a slight overestimate of correlated noise effects at low elevations.

To assess the effects of correlated noise, the SNR provided by the actual array is compared with that provided by the same antennas if arrayed in a configuration with infinitely long baselines, examining the ratio \( \beta_{\phi} / \beta_{\phi_0} \). This ratio depends not only on Earth–Jupiter distance, but, in a complicated way, on the array and source geometries and on hour angle. Figure 3 plots this ratio for a number of different geometries. In each plot the declination of Jupiter,\(^7\) the Earth–Jupiter distance, and the angular separation between the spacecraft and the center of Jupiter are held fixed. Each point on the plot then represents the value of \( \beta_{\phi} / \beta_{\phi_0} \) for a randomly chosen value of the hour angle, the orientation on the sky of Jupiter’s radiation belts, and the orientation of the spacecraft–Jupiter separation. Thus, the density of points on a portion of a plot provides an estimate of the likelihood, as a function of the hour angle, of obtaining a particular value of \( \beta_{\phi} / \beta_{\phi_0} \) for the given parameters.

Figures 4(a) and (b) summarize the results of the calculations shown in Fig. 3 plotting, as a function of spacecraft–Jupiter separation, the average (over hour angle and orientation) value of \( \beta_{\phi} / \beta_{\phi_0} \), as well as its minimum and maximum values. Thus, each plot in Fig. 3 is reduced to three points in Fig. 4—one on an average curve, one on a maximum curve, and one on a minimum curve. The following conclusions can be drawn from Fig. 4:

1. When Galileo is within ~75 arcsec of Jupiter, the average effect of correlated noise is a loss of SNR, which may be as large as 5 percent at zero separation.

2. When Galileo is more than 400 arcsec from Jupiter, the effects of correlated noise are negligible.

3. For separations in the range of 75 to 400 arcsec, the effects of correlated noise are small on average. However, the loss may be significant for certain geometries (as may be the enhancement). In this range of planet–spacecraft separations, careful calculations of correlated noise are necessary in a case where a 5-percent SNR loss would be critical.

The above conclusions refer specifically to an array consisting of the antennas listed in Table 1. Correlated noise effects are likely to be similar for arrays of similar antennas with similar baseline geometries, but quantitative calculations for other arrays have not been carried out. Because correlated noise causes significant SNR variations, and because these variations are highly geometry dependent, detailed calculations of these effects should be done for any situations in which a few tenths of a decibel of SNR are crucial. For most purposes involving Galileo and Jupiter, Eqs. (15), (B-6), and (B-9) should be suitable for such calculations.

\(^{7}\) A declination of ~21 deg, corresponding to that of the Galileo encounter in December 1995, has been used for Fig. 3. The results for other Jupiter declinations are different in detail but qualitatively very similar.
Acknowledgments

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Table 1. S-band parameters of antennas in proposed Canberra array.

<table>
<thead>
<tr>
<th>Antenna</th>
<th>Gain, K/Jy</th>
<th>System temperature, K</th>
<th>S-band beamwidth (FWHM), deg</th>
<th>Station coordinatesa</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>East, m</td>
</tr>
<tr>
<td>DSS 43</td>
<td>0.95</td>
<td>18.5</td>
<td>0.11</td>
<td>0.0</td>
</tr>
<tr>
<td>DSS 42</td>
<td>0.21</td>
<td>22.0</td>
<td>0.28</td>
<td>0.0003</td>
</tr>
<tr>
<td>DSS 45</td>
<td>0.16</td>
<td>38.0</td>
<td>0.23</td>
<td>-325.3907</td>
</tr>
<tr>
<td>DSS 34b</td>
<td>0.16</td>
<td>30.0</td>
<td>0.23</td>
<td>68.8</td>
</tr>
</tbody>
</table>

a Relative to DSS 43.
b Values for DSS 34 are approximate.
Fig. 1. The basic geometry of an array. Antennas $i$ and $k$ are located at, respectively, $r_i$ and $r_k$ with respect to the array reference point and are pointed toward the direction $\hat{s}_0$. It is clear from the diagram that the delay suffered by a wavefront from a direction $\hat{s}$ between antenna $i$ and antenna $k$ is proportional to $B_{ik} \cdot \hat{s}$.

Fig. 2. A simplified block diagram of an array signal path.
Fig. 3. The ratio of $\beta_{\theta}/\beta_{\phi_{oc}}$ for an array of antennas at the DSN Canberra complex observing Jupiter and a spacecraft at S-band, for a variety of array and source geometries. The declination of Jupiter = $-21.0$ deg and the array consists of DSS 43, DSS 42, DSS 45, and DSS 34.
Fig. 4. The average, minimum, and maximum values of $\beta_\phi / \beta_{\phi,\infty}$ obtained as source orientation is varied, plotted as a function of the separation between the spacecraft and the center of Jupiter, with Jupiter declination = -21.0 deg: (a) Earth–Jupiter distance = 4.2 AU and (b) Earth–Jupiter distance = 6.2 AU.
Appendix A

Basic Expressions

I. Incoming Signals

The voltage (as a function of time $t$ and sky frequency $\nu_{sky}$) induced in a single polarization channel at $r_i$, the focal point of the $i$th antenna, by a distant\(^6\) source (e.g., a planet or spacecraft) can be written

$$V_i(\nu_{sky}, t) = e^{i\phi_i} \sqrt{k_B G_i} \int \frac{d\tilde{s}}{\Delta s} f_i(\tilde{s} - \tilde{s}_0) I(\nu_{sky}, t, \tilde{s}) e^{i[2\pi \nu_{sky} t - (\tilde{s} \cdot \mathbf{r}_i/c) + \theta(\nu_{sky}, \tilde{s})]}$$  \hspace{1cm} (A-1)

In this expression, $k_B$ represents Boltzmann’s constant and $c$ is the speed of light; $G_i$ is the on-axis antenna gain, $f_i$ is the magnitude of the antenna’s field pattern, $\tilde{s}$ is a unit vector in the direction of the source, $I(\nu_{sky}, t, \tilde{s})$ is the amplitude of the electric field due to the source, $\theta(\nu_{sky}, \tilde{s})$ is a phase term (which for most astronomical sources can be assumed to be uncorrelated over $\nu_{sky}$ and $\tilde{s}$), and $\phi_i$ is an antenna-based phase shift (including atmospheric effects).

The voltage induced by system noise can be written

$$V_i^N(\nu_{sky}, t) = \sqrt{k_B T_i} e^{i[2\pi \nu_{sky} t + \theta_i^N(\nu_{sky}, t)]}$$ \hspace{1cm} (A-2)

where $T_i$ is the noise temperature and $\theta_i^N(\nu_{sky}, t)$ is a random phase that is uncorrelated over time and frequency intervals satisfying $\Delta \nu_{sky} \Delta t \gtrsim 1$. The total RF voltage can be written as the sum of terms due to the spacecraft, planet, and system noise:

$$V_i(\nu_{sky}, t) = V_i^{S/C}(\nu_{sky}, t) + V_i^P(\nu_{sky}, t) + V_i^N(\nu_{sky}, t)$$ \hspace{1cm} (A-3)

When the RF signal is mixed to baseband, it is phase shifted by $2\pi \nu_{lo} \tau_i^m$; this stops the fringes, allowing time averaging over a longer interval. An additional phase shift $-\phi_i^m$ can be inserted (attempt to) compensate for the antenna-based phase shifts $\phi_i$. Without this additional phase shift, the baseband voltage can be written

$$v_i(\nu, t) = e^{i[2\pi \nu_{lo} \tau_i^m]} V_i(\nu_{sky}, t)$$

$$= e^{i[2\pi \nu_{lo} \tau_i^m]} \left[ V_i^{S/C}(\nu_{sky}, t) + V_i^P(\nu_{sky}, t) + V_i^N(\nu_{sky}, t) \right]$$

$$= e^{i[2\pi \nu_{lo} \tau_i^m]} \sqrt{k_B G_i} \int \frac{d\tilde{s}}{\Delta s} f_i(\tilde{s} - \tilde{s}_0) e^{i[2\pi \nu_{sky} t - (\tilde{s} \cdot \mathbf{r}_i/c)]} I_{S/C}(\nu_{sky}, t, \tilde{s}) e^{i\theta_i^{S/C}(\nu_{sky}, \tilde{s})}$$

$$+ I_P(\nu_{sky}, t, \tilde{s}) e^{i\theta_i^P(\nu_{sky}, \tilde{s})} + e^{i[2\pi \nu_{lo} \tau_i^m]} \sqrt{k_B T_i} e^{i[2\pi \nu_{sky} t + \theta_i^N(\nu_{sky}, t)]}$$

$$= e^{i\phi_i} \sqrt{k_B G_i} \int \frac{d\tilde{s}}{\Delta s} f_i(\tilde{s} - \tilde{s}_0) e^{i[2\pi (t - (\tilde{s} \cdot \mathbf{r}_i/c)) + \nu_{lo} (\tau_i^m - (\tilde{s} \cdot \mathbf{r}_i/c))] I_{S/C}(\nu_{sky}, t, \tilde{s}) e^{i\theta_i^{S/C}(\nu_{sky}, \tilde{s})}$$

$$+ I_P(\nu_{sky}, t, \tilde{s}) e^{i\theta_i^P(\nu_{sky}, \tilde{s})} + \sqrt{k_B T_i} e^{i[2\pi (t + \nu_{lo} \tau_i^m)] + \theta_i^N(\nu_{sky}, t)]}$$ \hspace{1cm} (A-4)

\(^6\) Here, “distant” means a source sufficiently far away that, over the extent of the array, a plane wave approximation is valid.
where \( \nu = \nu_{sky} - \nu_{0} \) is the baseband frequency. Like the RF voltage, the baseband voltage can be written as the sum of terms due to the spacecraft, planet, and system noise:

\[
v_i(t, \nu) = v_i^{f/c}(t, \nu) + v_i^P(t, \nu) + v_i^N(t, \nu)
\]  
(A-5)

If the additional phase shift is inserted, the baseband signal has the form \( e^{-i\phi_{\nu}^*} v_i(t, \nu) \).

**II. Single-Dish Power**

For the \( i \)th antenna, the single-dish power can be written

\[
P_i = \left\langle \int_{0}^{\Delta \nu} d\nu |v_i(t, \nu)|^2 \right\rangle_T
\]

\[
= \left\langle \int_{0}^{\Delta \nu} d\nu \left[ v_i^{f/c}(t, \nu) v_i^{f/c*}(t, \nu) + v_i^P(t, \nu) v_i^P*(t, \nu) + v_i^N(t, \nu) v_i^N*(t, \nu) \right] \right\rangle_T
\]

\[
= \int_{0}^{\Delta \nu} d\nu \left[ v_i^{f/c}(t, \nu) v_i^{f/c*}(t, \nu) + v_i^P(t, \nu) v_i^P*(t, \nu) + v_i^N(t, \nu) v_i^N*(t, \nu) \right] \right\rangle_T
\]  
(A-6)

where the angle brackets denote a time average and \( T \) is the interval over which the time average is taken. Looking at each term separately and considering the planet term first,

\[
v_i^P(t, \nu) v_i^P*(t, \nu) = e^{i\phi} \sqrt{k_B G_i} \int d\hat{s} f_i(\hat{s} - \hat{s}_0) e^{i2\pi \nu t (\hat{s} - \hat{s}_0, c)} I_P(\nu + \nu_{0}, t, \hat{s}) e^{-i2\pi \nu t (\hat{s} - \hat{s}_0, c)} I_P(\nu + \nu_{0}, t, \hat{s})
\]

\[
\times e^{-i\phi} \sqrt{k_B G_i} \int d\hat{s}' f_i(\hat{s}' - \hat{s}_0) e^{-i2\pi \nu t (\hat{s}' - \hat{s}_0, c)} I_P(\nu + \nu_{0}, t, \hat{s}') e^{i2\pi \nu t (\hat{s}' - \hat{s}_0, c)} I_P(\nu + \nu_{0}, t, \hat{s}')
\]

\[
= k_B G_i \int d\hat{s} f_i(\hat{s} - \hat{s}_0) I_P(\nu + \nu_{0}, t, \hat{s}) \int d\hat{s}' \left\{ e^{-i2\pi \nu t (\hat{s}' - \hat{s}_0, c)} I_P(\nu + \nu_{0}, t, \hat{s}') \left\{ e^{i2\pi \nu t (\hat{s} - \hat{s}_0, c)} I_P(\nu + \nu_{0}, t, \hat{s}) \right\} \right\}
\]

\[
= k_B G_i \int d\hat{s} f_i^2(\hat{s} - \hat{s}_0) I_P^2(\nu + \nu_{0}, t, \hat{s})
\]

\[
= k_B G_i \int d\hat{s} f_i^2(\hat{s} - \hat{s}_0) I_P(\nu + \nu_{0}, t, \hat{s})
\]  
(A-7)
where the integral over \( \hat{s}' \) is nonzero only when \( \hat{s}' = \hat{s} \) because the radiation from the planet is spatially incoherent, and where the substitution \( I_P^c(\nu + \nu_0, t, \hat{s}) = I_P(\nu + \nu_0, t, \hat{s}) \) was made; \( I_P \) is the brightness distribution (flux per unit solid angle) of the planet. Similarly

\[
v_i^s/c(\nu, t) v_i^s/c^*(\nu, t) = k_B G_i \int d\hat{s}' f_i^2(\hat{s} - \hat{s}_0) I_{s/c}(\nu + \nu_0, t, \hat{s})
\]  

(A-8)

with \( I_{s/c} \) being the brightness distribution of the spacecraft (usually assumed to be pointlike), and

\[
v_i^N(\nu, t) v_i^N^*(\nu, t) = k_B T_i
\]

(A-9)

Compared to these terms, the cross terms are all small, as can be seen by considering

\[
v_i^s/c(\nu, t) v_i^P^*(\nu, t) = k_B G_i \int_{s/c} d\hat{s} f_i(\hat{s} - \hat{s}_0) I_{s/c}(\nu + \nu_0, t, \hat{s}) \int_{s/c} d\hat{s}' \left\{ e^{-i(2\pi/c)(\nu(\hat{s} - \hat{s}')) \cdot \hat{r}_i + \nu_0(\hat{s} - \hat{s}' \cdot \hat{r}_i)} \right\}
\]

\[
\times e^{i[\theta^s/c(\nu + \nu_0, \hat{s}) - \theta^P(\nu + \nu_0, \hat{s}')] f_i(\hat{s}' - \hat{s}_0) I_P(\nu + \nu_0, t, \hat{s}')}
\]

\[
\approx 0
\]

(A-10)

where the integral over \( \hat{s}' \) is small because \( \theta^s/c \) and \( \theta^P \) are uncorrelated [unlike in Eqs. (A-7) and (A-8), this integral is, on average, zero, even when \( \hat{s}' = \hat{s} \)].

One can therefore write

\[
P_i = \left\langle \int_0^{\Delta \nu} d\nu |v_i(\nu, t)|^2 \right\rangle_T
\]

\[
= \left\langle \int_0^{\Delta \nu} d\nu |v_i^s/c(\nu, t)|^2 + |v_i^P(\nu, t)|^2 + |v_i^N(\nu, t)|^2 \right\rangle_T
\]

\[
= P_i^s/c + P_i^P + P_i^N
\]  

(A-11)

It is assumed that the spacecraft is a point source located at \( \hat{s}_{s/c} \), transmitting a narrowband, polarized signal at a sky frequency \( \nu_{sky}^s/c \) with a total power \( P_{s/c} \). The brightness distribution seen by a receiver matched to the signal's polarization\(^9\) is

\[
I_{s/c}(\nu_{sky}, t, \hat{s}) = 2P_{s/c} \delta(\nu_{sky} - \nu_{sky}^s/c) \delta(\hat{s} - \hat{s}_{s/c})
\]  

(A-12)

(with \( \delta \) representing a Dirac delta function), so

\(^9\) The factor of 2 in Eq. (A-12) arises from the assumption of matched polarizations.
\[ P_{\text{eff}}^P = 2k_B f_i^2 (\tilde{s}_{s/e} - \tilde{s}_e) G_i \mathcal{P}_{s/e} \approx 2k_B G_i \mathcal{P}_{s/e} \]  

(A-13)

where it is assumed that \( f_i^2 (\tilde{s}_{s/e} - \tilde{s}_e) \approx 1 \) (i.e., the spacecraft is close to center of the array's field of view).

It is assumed that the planet is an extended, unpolarized, broadband source of total flux at the frequencies of interest of \( S_P \), which is constant in time and varies only slowly with frequency (i.e., it can be considered constant across the observing bandwidth \( \Delta \nu \)), so \( I_P(\nu, k, t, \hat{s}) = I_P(\hat{s}) \) and

\[ P_i^N = k_B T_i \Delta \nu \]  

(A-15)

### III. Cross-Correlation

The cross-correlation of the voltages from the \( i \)th and \( k \)th antennas is given by

\[ \rho_{ik} = \left( \int_{0}^{\Delta \nu} \int d\nu e^{i(\phi_{i}^N - \phi_{k}^N)} \right) \left( v_i(t, \nu, t + \tau_i^m) + v_i(t, \nu + \tau_i^m) + v_i(t, \nu + \tau_i^m) \right) \]  

(A-16)

By using the general form of the baseband signal [Eq. (A-4)] and because the signal is spatially incoherent, this becomes

\[ \rho_{ik} = \left( \int_{0}^{\Delta \nu} \int d\nu e^{i(\phi_{i}^N - \phi_{k}^N)} \right) \left( v_i(t, \nu, t + \tau_i^m) + v_i(t, \nu + \tau_i^m) + v_i(t, \nu + \tau_i^m) \right) \]  

(A-17)
This can be written as

$$\rho_{ik} = \rho_{ik}^s + \rho_{ik}^N$$  \hspace{1cm} (A-18)

where

$$\rho_{ik}^s = \left< \int_0^{\Delta\nu} d\nu e^{i(\phi_{i}^s/\tau_{i}^s)} v_i^s (\nu, t + \tau_{i}^m) v_k^s (\nu, t + \tau_{k}^m) \right>_T$$  \hspace{1cm} (A-19)

$$\rho_{ik}^N = \left< \int_0^{\Delta\nu} d\nu e^{i(\phi_{i}^N/\tau_{i}^N)} v_i^N (\nu, t + \tau_{i}^m) v_k^N (\nu, t + \tau_{k}^m) \right>_T$$  \hspace{1cm} (A-20)

These expressions are similar in structure to the expression for single-dish power [Eq. (A-6)]. Looking at \(\rho_{ik}^N\), one finds

$$v_i^N (\nu, t + \tau_{i}^m) v_k^N (\nu, t + \tau_{k}^m) = e^{i(\phi_{i}^N/\tau_{i}^N)} k_B \sqrt{G_i G_k} e^{i2\pi [\nu + \nu_i] (\tau_{i}^m - \tau_{k}^m)} \left[ \int d\tilde{s} \left\{ e^{-i2\pi [\nu + \nu_i] (\tilde{s} - \tilde{s}_0)} I_P (\tilde{s}) \times \int d\tilde{s}' f_k (\tilde{s}' - \tilde{s}_0) I_P (\tilde{s}') \right\} \right]$$

$$\times \int d\tilde{s} ' f_i (\tilde{s} - \tilde{s}_0) \left[ e^{i2\pi [\nu + \nu_i] (\tilde{s} - \tilde{s}_0)} e^{i(\theta_{i} - \theta_{k})/c} \right]$$

$$= e^{i(\phi_{i}^N/\tau_{i}^N)} k_B \sqrt{G_i G_k} e^{i2\pi [\nu + \nu_i] (\tilde{s} - \tilde{s}_0) / c} \left[ \int d\tilde{s} f_i (\tilde{s} - \tilde{s}_0) f_k (\tilde{s} - \tilde{s}_0) I_P (\tilde{s}) \right]$$

$$= e^{i(\phi_{i}^N/\tau_{i}^N)} k_B \sqrt{G_i G_k} \left[ \int d\tilde{s} e^{i2\pi [\nu + \nu_i] (\tilde{s} - \tilde{s}_0)} f_i (\tilde{s} - \tilde{s}_0) f_k (\tilde{s} - \tilde{s}_0) I_P (\tilde{s}) \right]$$

$$= e^{i(\phi_{i}^N/\tau_{i}^N)} k_B \sqrt{G_i G_k} \left[ \int d\tilde{s} e^{i2\pi [\nu + \nu_i] (\tilde{s} - \tilde{s}_0) B_{ik} / c} f_i (\tilde{s} - \tilde{s}_0) f_k (\tilde{s} - \tilde{s}_0) I_P (\tilde{s}) \right]$$  \hspace{1cm} (A-22)
where the substitutions \( \tau_{i}^{m} = s_{i} r_{i} / c \), \( \tau_{k}^{m} = s_{k} r_{k} / c \) and \( B_{ik} = r_{k} - r_{i} \) have been made.

Similarly
\[
v_{i}^{s/c}(\nu, t + \tau_{i}^{m}) v_{k}^{s/c}(\nu, t + \tau_{k}^{m}) = e^{i(\phi_{i} - \phi_{k})} k_{B} \sqrt{G_{i} G_{k}}
\]
\[
\times \int \int d\tilde{s} \ e^{i(2\pi[\nu + \nu_{0}][\tilde{s} - \tilde{s}_{0}] B_{ik} / c)} f_{i}(\tilde{s} - \tilde{s}_{0}) f_{k}(\tilde{s} - \tilde{s}_{0}) I_{s/c}(\tilde{s})
\]
(A-23)

In considering the effects of noise, it is assumed\(^{10}\) that Eq. (A-21) can be approximated as
\[
\rho_{ik}^{N} \approx \left\langle \int_{0}^{\Delta\nu} d\nu e^{i(\phi_{k}^{N} - \phi_{i}^{N})} v_{i}^{N}(\nu, t + \tau_{i}^{m}) v_{k}^{N}(\nu, t + \tau_{k}^{m}) \right\rangle_{T}
\]
\[
\approx k_{B} \sqrt{T_{1} T_{k}} \left\langle \int_{0}^{\Delta\nu} d\nu e^{i(\phi_{k}^{N} - \phi_{i}^{N})} e^{i(2\pi\nu[\tau_{k}^{m} - \tau_{i}^{m} + 2\nu_{0} \tau_{k}^{m} + \theta_{k}^{N}(\nu, t + \tau_{k}^{m}) - \theta_{i}^{N}(\nu, t + \tau_{k}^{m}))} \right\rangle_{T}
\]
\[
\approx k_{B} \sqrt{T_{1} T_{k}} \frac{\sqrt{\Delta\nu T}}{T} e^{i\theta_{ik}^{N}}
\]
\[
\approx k_{B} \sqrt{T_{1} T_{k}} \frac{\sqrt{\Delta\nu T}}{T} e^{i\theta_{ik}^{N}}
\]
(A-24)

where \( \theta_{ik}^{N} \) is a phase randomly distributed between 0 and 2\( \pi \), and where use has been made of the fact that \( \theta_{ik}^{N}(\nu_{ky}, t) \) is uncorrelated between antennas and over time and frequency intervals where \( \Delta\nu T > 1 \). Because the phases of \( \rho_{ik}^{N} \) are random, the double sum in Eq. (10) will be small. However, as discussed in Appendix C, noise terms become important when the process of phasing the array is considered.

Using Eq. (A-12) to substitute for \( I_{s/c} \),
\[
\rho_{ik}^{s/c} = 2P_{s/c} \left\langle \int_{0}^{\Delta\nu} d\nu e^{i(\nu - [\nu_{sy}^{s/c} - \nu_{0}])} e^{i(\phi_{k}^{N} - \phi_{i}^{N})} e^{i(\phi_{i} - \phi_{k})} k_{B} \sqrt{G_{i} G_{k}}
\]
\[
\times \int \int d\tilde{s} \ e^{i(2\pi[\nu + \nu_{0}][\tilde{s} - \tilde{s}_{0}] B_{ik} / c)} f_{i}(\tilde{s} - \tilde{s}_{0}) f_{k}(\tilde{s} - \tilde{s}_{0}) \delta(\tilde{s} - \tilde{s}_{s/c}) \right\rangle_{T}
\]
\[
= 2P_{s/c} k_{B} \sqrt{G_{i} G_{k}} e^{i(\phi_{i}^{N} - \phi_{k})} e^{i(2\pi\nu_{sy}^{s/c} B_{ik} / c)} f_{i}(\tilde{s}_{s/c} - \tilde{s}_{0}) f_{k}(\tilde{s}_{s/c} - \tilde{s}_{0})
\]
(A-25)

\(^{10}\) This assumption is not valid if the planet or the spacecraft contributes significantly to the system temperature; in that case, the cross terms in Eq. (A-21) will be non-negligible. However, they can be treated in a manner similar to the \( \nu^{N} \) term.
Note that \( \hat{s}_{fc} - \hat{s}_0 \) must be small enough that the quantity \( \nu_{sky}^P [\hat{s}_{fc} - \hat{s}_0] \cdot B_{ik}/c \) does not change significantly over the interval of the time averaging.

Similarly,

\[
\rho_{ik}^P = k_B \sqrt{G_i G_k} e^{i(\delta \phi_i - \delta \phi_k)} \left\langle \int \int d \hat{s} f_i(\hat{s} - \hat{s}_0) f_k(\hat{s} - \hat{s}_0) I_P(\hat{s}) \int_0^{\Delta \nu} d \nu e^{i(2 \pi \nu B_{ik} (\hat{s} - \hat{s}_0)/c)} \right\rangle
\]

(A-26)

At this point, it is useful to define the quantity

\[
\mathcal{F}_{ik}^P \equiv \frac{1}{\Delta \nu S_P} \int \int d \hat{s} e^{i(2 \pi \nu B_{ik} (\hat{s} - \hat{s}_0)/c)} f_i(\hat{s} - \hat{s}_0) f_k(\hat{s} - \hat{s}_0) I_P(\hat{s}) \int_0^{\Delta \nu} d \nu e^{i(2 \pi \nu B_{ik} (\hat{s} - \hat{s}_0)/c)}
\]

(A-27)

so that

\[
\rho_{ik}^P = \Delta \nu k_B S_P \sqrt{G_i G_k} e^{i(\delta \phi_i - \delta \phi_k)} \mathcal{F}_{ik}^P
\]

(A-28)

\( \mathcal{F}_{ik}^P \) is a dimensionless complex quantity that depends only on the array geometry and the source structure and whose magnitude is always less than or equal to unity (\( |\mathcal{F}_{ik}^P| \rightarrow 1 \) in the short baseline limit, and \( |\mathcal{F}_{ik}^P| \rightarrow 0 \) in the long baseline limit).

It has been implicitly assumed that the bandpass is rectangular, in which case the integral over \( \nu \) can be simplified:

\[
\int_0^{\Delta \nu} d \nu e^{i(2 \pi \nu B_{ik} (\hat{s} - \hat{s}_0)/c)} = e^{i(\pi \Delta \nu B_{ik} [\hat{s} - \hat{s}_0]/c)} \text{sinc} \left( \frac{\Delta \nu}{c} B_{ik} [\hat{s} - \hat{s}_0] \right)
\]

(A-29)

where \( \text{sinc}(x) = \sin(\pi x)/\pi x \). This term, often referred to as the delay beam, introduces a phase shift and lowers the correlation amplitude for \( \hat{s} \neq \hat{s}_0 \); both these effects are more pronounced for larger bandwidths \( \Delta \nu \).

One can therefore write

\[
\mathcal{F}_{ik}^P = \frac{1}{\Delta \nu S_P} \int \int d \hat{s} e^{i(2 \pi \nu_{sky} B_{ik} (\hat{s} - \hat{s}_0)/c)} f_i(\hat{s} - \hat{s}_0) f_k(\hat{s} - \hat{s}_0) I_P(\hat{s}) \text{sinc} \left( \frac{\Delta \nu}{c} B_{ik} [\hat{s} - \hat{s}_0] \right)
\]

(A-30)

where \( \nu_{sky} = \nu_{ic} + \Delta \nu/2 \).

If the quantities \( f_i, f_k \) and \( \text{sinc}(\Delta \nu B_{ik} [\hat{s} - \hat{s}_0]/c) \) do not vary greatly over the extent of the planet, one can make the further simplification

\[
\mathcal{F}_{ik}^P = \frac{1}{\Delta \nu S_P} \bar{f}_{ip} \bar{f}_{kp} \bar{f}_{ik} \int \int d \hat{s} e^{i(2 \pi \nu_{sky} B_{ik} (\hat{s} - \hat{s}_0)/c)} I_P(\hat{s})
\]

(A-31)

where \( \bar{f}_{ip}, \bar{f}_{kp}, \bar{f}_{ik} \) are suitably weighted averages of \( f_i \), \( f_k \) and \( \text{sinc}(\Delta \nu B_{ik} [\hat{s} - \hat{s}_0]/c) \), respectively.
Appendix B

Jupiter Model

At centimeter wavelengths, Jupiter is not a simple thermal disk; there is significant synchrotron emission from the radiation belts and the resulting flux distribution is quite complicated [6,7]. For the purposes of this article, Jupiter can be modelled as the sum of three components: two circular (2-dimensional) Gaussian components (representing the radiation belts) and a uniform central disk. This is a simple model to work with because Eq. (A-27) can be integrated analytically for each of these components.

If the brightness distribution of a source is radially symmetric about the point \( \hat{s}_C \), small compared to the delay beam, and small compared to the primary beam of any antenna, Eq. (A-27) can be written

\[
\mathcal{F}_{ik} = \frac{2\hat{f}_i \hat{f}_k}{S} \hat{f}_{\Delta \nu_{ik}} e^{i(2\pi \nu_{sk} B_{ik} [\hat{s}_C - \hat{s}_0]/c)} \int_0^\infty r \ dr \ I(r) J_0 \left( \frac{2\pi \nu_{sk} B_{ik} r}{c} \right)
\]

where \( r = |\hat{s} - \hat{s}_C| \); \( \hat{f}_{\Delta \nu_{ik}} \) is the value of the delay beam at the source; \( \hat{f}_i, \hat{f}_k \) are the average values of the antenna field pattern at the source; for the \( i \)th and \( k \)th antenna; \( S \) is the total flux of the source; \( I(r) \) is the source’s radially symmetric brightness distribution; and \( J_0 \) is a zeroth-order Bessel function.

For a uniform disk of angular radius \( R_D \) centered at \( \hat{s}_C \),

\[
I(r) = \frac{S}{\pi R_D^2} \quad r \leq R_D
\]

\[
= 0 \quad r > R_D
\]

\[
\mathcal{F}^D_{ik} = \hat{f}_i \hat{f}_k \hat{f}_{\Delta \nu_{ik}} e^{i(2\pi \nu_{sk} B_{ik} [\hat{s}_C - \hat{s}_0]/c)} \frac{e^{i(2\pi \nu_{sk} B_{ik} R_D/c)}}{\pi \nu_{sk} B_{ik} R_D}
\]

(\ref{eq:uniform_disk})

For a circular Gaussian of total flux \( S \) and characteristic size (1/e radius) \( R_G \), centered at \( \hat{s}_C \),

\[
I(r) = \frac{S}{\pi R_G^2} e^{-r^2/R_G^2}
\]

(\ref{eq:gaussian})

and Eq. (B-1) becomes

\[
\mathcal{F}^G_{ik} = \hat{f}_i \hat{f}_k \hat{f}_{\Delta \nu_{ik}} e^{i(2\pi \nu_{sk} B_{ik} [\hat{s}_C - \hat{s}_0]/c)} \frac{e^{-(\pi \nu_{sk} B_{ik} R_G/c)^2}}{\pi}
\]

(\ref{eq:gaussian})

Jupiter’s brightness distribution is modelled as the sum of a uniform central disk of radius \( R_J \), centered at \( \hat{s}_J \), and two circular Gaussians of characteristic size \( R_B \), representing the radiation belts, which are offset from the center of the disk by \( \pm \Delta s_B \). The flux of the central disk is written as \( F_D S_J \), where \( S_J \) is the total flux of Jupiter and \( F_D \) is the fraction of that flux in the disk; the integrated flux of each wing component can be written as \((1 - F_D) S_J / 2\). Using this model with Eqs. (\ref{eq:uniform_disk}) and (\ref{eq:gaussian}),

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\[ F_{ik}^I = \int_I \int_J \int_k \int_{\Delta\nu} e^{i(2\pi\nu_{sk} B_{ik} R_J/c)} \left[ F_D \frac{c J_1(2\pi\nu_{sk} B_{ik} R_J/c)}{\pi \nu_{sk} B_{ik} R_J} \right. \\
+ (1 - F_D) \frac{e^{-(\pi\nu_{sk} B_{ik} R_B/c)^2}}{\pi} \cos \left( \frac{2\pi\nu_{sk} B_{ik} \Delta s_B}{c} \right) \right] \]

(B-6)

If the above expression and Eq. (C-1), the condition that the array is properly phased on the spacecraft, are substituted into Eq. (A-28), one gets, for the cross-correlation due to Jupiter in a phased array,

\[ \rho_{ik}^J = \Delta \nu_{jk} S_J \sqrt{G_i G_k} \int_J \int_k \int_{\Delta\nu} e^{i(2\pi\nu_{sk} B_{ik} R_J/c)} \left[ F_D \frac{c J_1(2\pi\nu_{sk} B_{ik} R_J/c)}{\pi \nu_{sk} B_{ik} R_J} \right. \\
+ (1 - F_D) \frac{e^{-(\pi\nu_{sk} B_{ik} R_B/c)^2}}{\pi} \cos \left( \frac{2\pi\nu_{sk} B_{ik} \Delta s_B}{c} \right) \right] \]

(B-7)

In the case where \( \nu_{sk} = \nu_{sk} \) and/or \( \nu_{sk}^J = \nu_{sk}^J \), this simplifies to

\[ \rho_{ik}^J = \Delta \nu_{jk} S_J \sqrt{G_i G_k} \int_J \int_k \int_{\Delta\nu} e^{i(2\pi\nu_{sk} B_{ik} \Delta s_J/c)} \left[ F_D \frac{c J_1(2\pi\nu_{sk} B_{ik} R_J/c)}{\pi \nu_{sk} B_{ik} R_J} \right. \\
+ (1 - F_D) \frac{e^{-(\pi\nu_{sk} B_{ik} R_B/c)^2}}{\pi} \cos \left( \frac{2\pi\nu_{sk} B_{ik} \Delta s_B}{c} \right) \right] \]

(B-8)

Both the total flux \( S_J \) and the flux distribution are functions of frequency and Earth–Jupiter distance. From the data given in [5], the model at S-band (2.3 GHz) used in this article is

\[ S_J = 6.3 \left( \frac{4.04 \text{AU}}{d} \right)^2 \text{Jy} \]

\[ F_D = 0.3 \]

\[ R_J = 24.3 \frac{4.04 \text{AU}}{d} \text{arcsec} \]

\[ |\Delta s_B| \approx 2 R_J \]

\[ R_B \approx 1.3 R_J \]

(B-9)

At conjunction, \( d = 6.2 \text{ AU}, S_J = 2.7 \text{ Jy}, \) and \( R_J = 15.8 \text{ arcsec}; \) at opposition, \( d = 4.2 \text{ AU}, S_J = 5.8 \text{ Jy}, \) and \( R_J = 23.4 \text{ arcsec}. \)

Figure B-1(a) shows a map of Jupiter from [6] at 2.9 GHz (10.4 cm), while Fig. B-1(b) shows a higher resolution map from [7] at 1.4 GHz. Note that in Fig. B-1(b), the disk has not been removed. Figure (B-2) shows the model of Jupiter used in this article in the same format as Fig. B-1(a).
Fig. B-1. A map of the brightness distribution of Jupiter: (a) at 2.9 GHz with a 260-K disk component removed and 20-K contour intervals (from [6]) and (b) at 1.4 GHz, including disk component with contour intervals at 2, 5, 10, 20, 30, 55, 40, 60, 60, 70, 80, and 90 percent (from [7]).

Fig. B-2. A model of Jupiter's brightness distribution at S-band [see Eq. (B-9)], plotted in the same format as Fig. B-1(a) with disk component removed.
Appendix C
Phasing

I. The Ideal Case

The SNR on the spacecraft is maximized when the terms in the double sum in Eq. (8) add in phase. This can be (approximately) arranged with the appropriate choice of model phases $\phi_i^m$. By using Eq. (A-25), $\rho_{ik}^{s/c}$ will have zero phase for all $i,k$ when

$$\delta \phi_k - \delta \phi_i = \phi_k - \phi_i^m - \phi_i + \phi_i^m = \frac{2 \pi \nu_{sky}^{s/c}}{c} B_{ik} \cdot (\hat{s}_{i/c} - \hat{s}_0)$$

(C-1)

The above relation can be satisfied if, for all $i = 1, \ldots N,$

$$\phi_i^m = \phi_i + \frac{2 \pi \nu_{sky}^{s/c}}{c} B_{ir} \cdot (\hat{s}_{i/c} - \hat{s}_0)$$

(C-2)

where $r$ represents an arbitrarily chosen reference antenna.

The values of $\phi_i$ are not known a priori, but there are $N(N-1)/2$ independent measurements of the phase of the correlation function that can be used to fit for the $N$ model phases $\phi_i^m$ (see [8]). Note that since $\delta \phi_i, \delta \phi_k$ include media effects as well as phase shifts in each antenna's signal path, they vary on timescales of seconds to minutes, so that the model phases must be recalculated at least that frequently to prevent degradation of the signal.

II. The Effects of Random Noise and a Background Planet

The above discussion ignores the contributions of random noise and a background planet to the observed phases. These complicate the process of solving for model phases, since what actually can be observed is the phase of the total correlation $\rho_{ik}$, not the phase of $\rho_{ik}^{s/c}$, the correlation due to the spacecraft signal.

The phase $\Phi_{ik}^\rho$ of the correlation function is given by

$$\tan \Phi_{ik}^\rho = \frac{\text{Im}(\rho_{ik})}{\text{Re}(\rho_{ik})} = \frac{\text{Im}(\rho_{ik}^{s/c}) + \text{Im}(\rho_{ik}^N)}{\text{Re}(\rho_{ik}^{s/c}) + \text{Re}(\rho_{ik}^N)}$$

(C-3)

where Re and Im represent, respectively, the real and imaginary parts of the complex function. The simplest phasing algorithms make the approximation that the spacecraft signal is the dominant contribution to $\rho_{ik}$, i.e.,

$$\tan \Phi_{ik}^\rho = \frac{\text{Im}(\rho_{ik})}{\text{Re}(\rho_{ik})} \approx \frac{\text{Im}(\rho_{ik}^{s/c})}{\text{Re}(\rho_{ik}^{s/c})}$$

(C-4)

This approximation requires that $|\rho_{ik}^{s/c}| \gg |\rho_{ik}^P|$ and $|\rho_{ik}^{s/c}| \gg |\rho_{ik}^N|$ for all baselines $B_{ik}$. Using Eqs. (A-24), (A-25), and (A-28), this implies

$$P_{s/c} \gg \frac{\Delta \nu S_P}{2} |F_{ik}^P|$$

(C-5)

and

$$P_{s/c} \gg \frac{1}{2} \sqrt{\frac{T_i}{G_i} \frac{T_k}{G_k} \Delta \nu}$$

(C-6)

In theory, the first of these conditions can be circumvented by modelling the contribution of $\rho_{ik}^P$ to the phase of $\rho_{ik}$ in the phasing algorithm. The second condition is more fundamental and can only be dealt with by decreasing the observing bandwidth (if the spacecraft is a narrowband source) or increasing the time over which the correlation is averaged.
Appendix D

Weighting Factors

The weighting factors \( W_i \) should be chosen so as to maximize \( \beta_\phi \), which is given by Eq. (15). Such \( W_i \) will satisfy the condition

\[
\frac{d\beta_\phi}{dW_j} = 0
\]

for all \( j = 1, \ldots, N \). Since one of the weighting factors can be chosen arbitrarily, this condition amounts to \( N - 1 \) equations for \( N - 1 \) variables.

If the effects of the planet can be ignored (i.e., \( S_p \approx 0 \)), Eq. (15) reduces to

\[
\beta_\phi = \frac{\left[ \sum_{i=1}^{N} W_i^2 G_i + \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k} \right]}{\sum_{i=1}^{N} W_i^2 T_i}
\]

and Eq. (D-1) becomes

\[
\frac{d\beta_\phi}{dW_j} = 0 = \frac{\left[ \sum_{i=1}^{N} W_i^2 T_i \right] \left[ 2W_j G_j + \sum_{k \neq j}^{N} W_k \sqrt{G_j G_k} \right] - 2W_j T_j \left[ \sum_{i=1}^{N} W_i^2 G_i + \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k} \right]}{\left[ \sum_{i=1}^{N} W_i^2 T_i \right]^2}
\]

It is not terribly difficult to show that this is satisfied for

\[
W_i = \frac{\sqrt{G_i}}{T_i}
\]

If the contribution of the planet to the system temperature at each antenna is accounted for, but the correlated noise terms are ignored, the optimal weighting is

\[
W_i = \frac{\sqrt{G_i}}{T_i + G_i S_p}
\]

This is appropriate for an extremely extended array. If correlated noise terms are included, it is difficult (perhaps impossible) to solve for \( W_i \) in closed form, but it can be done numerically. For the array configuration at the DSN complex in Australia, the weights obtained, including the effects of correlated noise, differ by only a few percent from those given by Eq. (D-5), and they have a negligible effect (\( \lesssim 0.5 \) percent) on the values of \( \beta_\phi \). In fact, though the values of \( W_i \) obtained, including the effects of correlated noise, often differ by \( \sim 10 \) percent from those obtained from Eq. (D-4), the resulting values of \( \beta_\phi \) differ by \( \lesssim 1 \) percent. Therefore, throughout this article, Eq. (D-4) is used to calculate weighting factors.
References


