Analysis of Command Detector In-Lock Monitoring

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We have investigated a command detector in-lock monitoring strategy that uses N *estimates of* $(SNR)^{1/2}$ *each composed of* M *samples from both data and error channel outputs. The detector recognizes only two states (in-lock and out-of-lock) and indicates state transition when* N *successive* $(SNR)^{1/2}$ *estimates violate a threshold. We give the probabilities of indicating in-lock given the detector is out-of-lock and out-of-lock given in-lock as a function of threshold for* (N,M) = *(1,10), (2,5), (5,2), (10,1). From these probabilities a threshold compatible with design requirements can be determined.*

I. Introduction

A command detector being developed for NASA uses signal-to-noise ratio (SNR) estimates to monitor operations of the detector. An important monitoring function is to determine whether or not the detector is in-lock. The purpose of this article is to investigate how well the monitoring function can be performed using a particular set of strategies that can be relatively easily implemented.

In Section II we will detail the particular set of strategies we wish to pursue. We will measure their performance in terms of the conditioned probabilities of indicating in-lock, given the detector is out-of-lock, and indicating out-of-lock, given in-lock. In Section III, we will develop expressions for those conditional probabilities whose evaluation requires numerical integration. In Section IV, we will present the results of the numerical evaluations for strategies that appear most relevant to present design plans for the command detector.

II. Statement of Command Detector Strategy

In a previous DSN article (Ref. 1), we analyzed a method for estimating the square root of the SNR rather than the SNR itself. This method involved obtaining $(SNR)^{1/2}$ from the ratio of an average of M absolute values of data (in-phase) channel integrated outputs to an average of M absolute values of error (quadrature) channel integrated outputs. The strategy for using this $(SNR)^{1/2}$ estimate as an in-lock indicator is the following: We consider the command detector as being in one of two states: in-lock or out-of-lock. Initially, before the command is acquired, the detector is out-of-lock. We define transition to the in-lock state as occurring after N consecutive independent $(SNR)^{1/2}$ estimates are each above a threshold T_0. Once the in-lock state is reached, we define transition to the out-of-lock state as occurring after N consecutive independent $(SNR)^{1/2}$ estimates are each below a threshold T_0. Since NM samples of both data and error channels are required for the in-lock monitoring function, we will consider schemes for which the product of N and M is constant. The performance of a set with constant NM will be determined by considering the conditional probabilities of indicating in-lock, given out-of-lock, and of indicating out-of-lock, given in-lock.

III. Derivation of Expressions for the Conditional Probabilities

A. Expression for a Useful Probability

The following have been demonstrated in a previous DSN Progress Report (Ref. 1). The in-phase channel integrated output samples X_i are independent random variables identically distributed with probability density

$$P_X(\alpha) = \begin{cases} \dfrac{1}{(2\pi\sigma^2)^{1/2}} \left\{ \exp\left[-\dfrac{1}{2\sigma^2}(\alpha - AT)^2 \right] \right. \\ \left. + \exp\left[-\dfrac{1}{2\sigma^2}(\alpha + AT)^2 \right] \right\}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases} \tag{1}$$

where A is the signal amplitude, T is the integration time, and $\sigma^2 = N_0 T/2$ with $N_0/2$ the two-sided power spectral noise density. The error channel integrated output samples Y_i likewise are independent identically distributed random variables with probability density given by Eq. (1) with $A = 0$. The actual SNR is

$$\mathrm{SNR} = ½(AT/\sigma)^2 \tag{2}$$

and the command detector estimate W of (SNR)½ is

$$W = (\pi)^{-1/2} \frac{1}{M} \sum_{i=1}^{M} X_i \left[\frac{1}{M} \sum_{i=1}^{M} Y_i \right]^{-1} \tag{3}$$

For the present investigation we need to calculate the probability that $W < T_0$ when actual SNR is, say, S_0^2. Define random variables ν,Δ equal to

$$\frac{1}{M}\sum_{i=1}^{M} X_i, \; \frac{1}{M}\sum_{i=1}^{M} Y_i$$

respectively. Then, the needed probability $P(M,T_0,S_0^2)$ is

$$P(M,T_0,S_0^2) \equiv \int\int d\alpha_1\, d\alpha_2\, P_\nu(\alpha_1)\, P_\Delta(\alpha_2)\, \theta\left(\sqrt{\pi}\, T_0 - \frac{\alpha_1}{\alpha_2} \right) \tag{4}$$

where the θ-function, which is one if its argument is greater or equal to zero, and zero otherwise, has the integral representation

$$\theta(\beta) = \lim_{\epsilon \to 0} -\frac{1}{2\pi i} \int_{-\infty}^{-\infty} \frac{dY}{Y + i\epsilon} \exp(-i\beta Y) \tag{5}$$

Substituting this into Eq. (3) and interchanging order of integration gives

$$P(M,T_0,S_0^2) = \lim_{\epsilon \to 0} -\frac{1}{2\pi i} \int_{-\infty}^{-\infty} \frac{dY}{Y + i\epsilon} M_\nu(Y) \left[M_\Delta(\sqrt{\pi}\, T_0 Y) \right]^* \tag{6}$$

where $M_\nu(\alpha)$, $M_\Delta(\alpha)$ are the characteristic functions of ν,Δ respectively. Using Eq. (1) and the independence of the X_i, we have

$$M_\nu(\alpha) = \left\{ \frac{1}{2} \exp\left(-\frac{\sigma^2\alpha^2}{2M^2} \right) \left[\exp\left(i\frac{AT\alpha}{M} \right) \mathrm{erfc}\left(-\frac{AT}{\sqrt{2}\sigma} + i\frac{\sigma\alpha}{\sqrt{2}M} \right) + \exp\left(-i\frac{AT\alpha}{M} \right) \mathrm{erfc}\left(\frac{AT}{\sqrt{2}\sigma} + \frac{i\sigma\alpha}{\sqrt{2}M} \right) \right] \right\}^M \tag{7}$$

where erfc (x) is the complement of the error function:

$$\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

We obtain $M_\Delta(\alpha)$ from $M_\nu(\alpha)$ when $A = 0$, so

$$M_\Delta(\alpha) = \left\{ \exp\left(-\frac{\sigma^2\alpha^2}{2M^2} \right) \mathrm{erfc}\left(i\frac{\sigma\alpha}{\sqrt{2}M} \right) \right\}^M \tag{8}$$

With a change of variable $u = [(1 + \pi T_0^2)/2M]^{1/2} \sigma Y$ and using the symmetry relations of erfc (x) (Ref. 2) we have, from Eq. (6):

$$P(M,T_0,S_0^2) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{du}{u} \exp(-u^2) \times Im[g(u)\, \mathrm{erfc}(-i\sqrt{\pi}\gamma T_0 u)]^M \tag{9}$$

where Im means "imaginary part of," $\gamma = [M(1 + \pi T_0^2)]^{1/2}$, and we have defined the function $g(u)$:

$$g(u) \equiv \exp(-i\,2\gamma S_0 u) - i \times Im[\exp(-i\,2\gamma S_0 u)\, \mathrm{erfc}(S_0 - i\gamma u)] \tag{10}$$

Eqs. (9) and (10) use S_0^2 for the actual SNR $= ½(AT/\sigma)^2$.

B. Probabilities Involved in In-Lock Indicator Performance

In Eq. (9) we give an expression for the probability that the $(\mathrm{SNR})^{1/2}$ estimate W is less than T_0 given actual SNR equals S_0^2. To evaluate the schemes proposed in Section II we need to relate actual SNR to the in-lock and out-of-lock states. We will assume the in-lock state exists when actual SNR is above command detector design point of $S_0^2 = 11.22$ (corresponding to 10.5 dB) and the out-of-lock state exists when actual SNR equals $S_0^2 = 0$. With these assumptions, we have for the conditional probability of indicating in-lock given out-of-lock:

$$P(IN\,|\,OUT) = [1 - P(M,T_0,0)]^N \qquad (11)$$

and for the conditional probability of indicating out-of-lock given in-lock:

$$P(OUT\,|\,IN) \leq [P(M,T_0,11.22)]^N \qquad (12)$$

IV. Results and Conclusions

Present design plans for the command detector commit 10 data and error output samples for in-lock indication monitoring. This means the product NM must be 10. There are, of course, four ways to accomplish this: $(N,M) = (1,10)$, $(2,5)$, $(5,2)$, and $(10,1)$. We wish, therefore, to have the probabilities of Eqs. (11) and (12) for these four possibilities as a function of T_0.

The required probabilities require numerical evaluation of Eq. (9). While conceptually straightforward, this integration requires some care, since the integrand is oscillatory, so the integral achieves a small value from cancelling contributions. For example, Eq. (9) immediately suggests Hermite integration. However, the required values of N and M cause oscillations of the integrand too rapid to be reliably approximated by Hermite polynomials of degree ≤ 20, which is the highest degree with tabulated roots and weights (Ref. 3). Instead, we used the transformation $u = \tan^{-1} v$, divided the interval $[0, \pi/2]$ into equal intervals, and used 8-point Gaussian quadrature in each interval. The value of the integrals was insensitive to the number of partitioning intervals when the number exceeded 175. For the erfc functions of complex argument we used approximations depending upon the magnitude of the argument[1] to insure accuracy and rapid convergence of the summations involved. Since we could calculate analytically

$$P(1,T_0,0) = \frac{2}{\pi} \tan^{-1} (\sqrt{\pi}\, T_0),$$

we checked the numerical integration in this case and obtained agreement to 5 significant decimal figures for $P(IN\,|\,OUT) = [1 - P(1,T_0,0)]^{10}$.

The results of numerical evaluation of Eqs. (11) and (12) for the four possibilities as functions of T_0^2 (the equivalent threshold SNR) are presented in Fig. 1. From these curves one can establish a threshold to meet design requirements. If, for example, the requirements are $P(IN\,|\,OUT) < 10^{-3}$ and $P(OUT\,|\,IN) < 10^{-9}$, then we see that $(N,M) = (5,2)$ with $1 < T_0^2 < 1.7$ will be adequate.

[1]When the complex argument of the erfc-function had a squared magnitude less than 26 we used the approximation of Eq. 7.1.29 on p. 299 of Ref. 2. When the squared magnitude was greater than 26 we utilized the relation between the erfc-function and the confluent hypergeometric function to develop a 10-term asymptotic expansion for erfc derived from Eq. 13.5.1 of Ref. 2. The value 26 was chosen to insure rapid convergence of summations involved. Both representations of erfc agreed to 5 significant decimal figures for squared magnitudes between 20 and 26.

References

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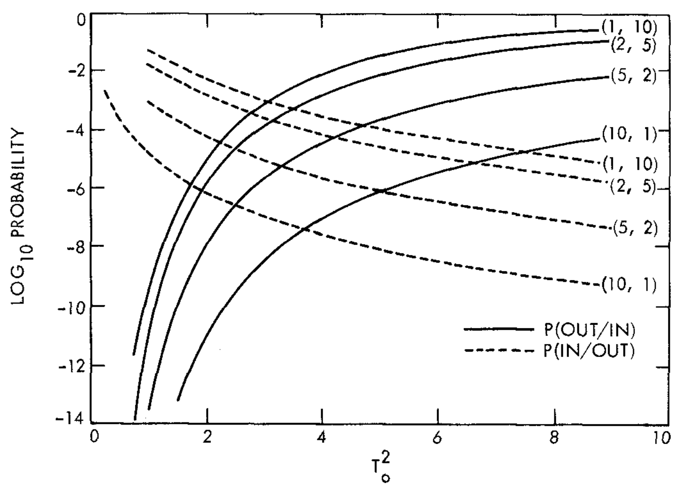

Fig. 1. Plot of conditional probabilities as a function of threshold for strategies (*N*,*M*) = (1,10), (2,5), (5,2), (10,1). *N* is the number of estimates of $(SNR)^{1/2}$ and *M* is the number of samples in each estimate.