The Effect of Direct Current Bias in the Computation of Power Spectra

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We determine the effect of dc bias in the approximate computation of spectra of Gaussian processes by hard limiting.

I. Introduction

The spectral density $S(\omega), |\omega| \leq \pi$, of a stationary Gaussian process $\{x_n\}, -\infty < k < \infty$, of mean zero, can be expressed in terms of the covariances $R_x(k) = E(x_n x_{n+k})$ by

$$S(\omega) = R_x(0) + 2 \sum_{k=1}^{\infty} R_x(k) \cos k\omega. \tag{1}$$

If we put

$$y_i = \begin{cases} +1, & x_i \geq 0, \\ -1, & x_i < 0, \end{cases} \tag{2}$$

Then it is known (Ref. 1) that $R_y(k) = E(y_n y_{n+k})$ satisfies the relation

$$R_y(k) = R_x(0) \sin \left[ \frac{\pi}{2} R_x(k) \right]. \tag{3}$$

In practice, the series in (1) is truncated, and the correlations $R_y(k)$ are estimated from samples of finite size, which leads to random errors in the evaluation of $S(\omega)$. These errors will not be considered here. Neglecting them, we use the variables $\{y_i\}$ to estimate $R_y(k)$, then apply (3) and (1) to get $S(\omega)$. This leads to a saving in computation time over the direct estimation of $R_x(k)$ from $\{x_n\}$, since these estimates require a large
number of arithmetic operations. The variance \( \sigma^2 = R_s(0) = E(x_i^2) \), if needed, must be estimated separately. This variance, which enters \( S(\omega) \) as a scale factor, is often unimportant, because it can be affected by many extraneous factors.

We consider the following problem: Suppose that an unknown bias \( a \) is added to the Gaussian process \( \{x_k\} \), so that we get \( \{x_k + a\}, -\infty < k < \infty \). The formula (2) cannot be applied if \( a \) is unknown. However, we can take

\[
y_i = \begin{cases} +1, & x_i + a \geq 0, \\ -1, & x_i + a < 0. \end{cases}
\]

Then we can estimate the numbers \( E_k = E(y_n y_{n+k}) \) (no longer correlations) take

\[
R_*^s (k) = \sin \left( \frac{\pi}{2} E_k \right),
\]

in analogy with (3), and form

\[
S^*(\omega) = R_*^s (0) + 2 \sum_{k=1}^{\infty} R_*^s (k) \cos k \omega.
\]

If \( a = 0 \), \( S^*(\omega) \) is \( S(\omega)/\sigma^2 \). If \( a \neq 0 \), \( S^*(\omega) \) approximates \( S(\omega) \) in some sense. We shall investigate how close this approximation is.

To find \( S(\omega) \) exactly, (3) needs to be replaced by a formula involving a non-elementary function which depends on \( a/\sigma \). This is not practical for real-time computations. Hence we consider the above approximation.

**II. Formulas**

Let the Gaussian random variables \( x_k \) have variance \( \sigma^2 \). Then the value of \( E_k = E(y_n y_{n+k}) \) depends only on \( \rho_k = R_s(k)/\sigma^2 \), and can be written as

\[
E_k = f(\rho_k),
\]

where

\[
f(z) = \frac{1}{2\pi \sqrt{1 - z^2}} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y(t_1) y(t_2) \exp \left[ \frac{t_1^2 + t_2^2 - 2zt_1 t_2}{2(1 - z^2)} \right] dt_1 dt_2,
\]

with

\[
y(t) = \begin{cases} +1, & t \geq -a/\sigma, \\ -1, & t < -a/\sigma. \end{cases}
\]

This follows if we evaluate \( E_k \) directly as the expectation of a function of \( x_n \) and \( x_{n+k} \), by integrating over the bivariate Gaussian distribution (Ref. 2).

If we differentiate (8) with respect to \( z \), the resulting integral can be simplified by integration by parts, and we get

\[
f'(z) = \frac{2}{\pi \sqrt{1 - z^2}} \exp \left( -\frac{a^2/\sigma^2}{1 + z^2} \right).
\]

Integrate this equation from 0 to \( z \). Using the variable of integration \( u = \sin^{-1} z \),

\[
f(z) - f(0) = \frac{2}{\pi} \int_{0}^{\sin^{-1} z} \exp \left( -\frac{a^2/\sigma^2}{1 + u^2} \right) du.
\]

To evaluate \( f(0) \), note that at \( z = 1 \), the distribution in (8) is concentrated on the line \( t_1 = t_2 \). Then \( f(1) = E(y(t_1)^2) = 1 \). Hence, putting \( z = 1 \) in (9),

\[
1 - f(0) = \frac{2}{\pi} \int_{0}^{\pi/2} \exp \left( -\frac{a^2/\sigma^2}{1 + u} \right) du.
\]

By numerical integration of the integral in (9), going from \( z = 0 \) to \( z = 1 \) and to \( z = -1 \), tables of \( f(z) \) were obtained for various values of \( a/\sigma \). Some of these are plotted in Fig. 1.

If we plot \( [\sin(\pi/2)f(z)] \) instead of \( f(z) \), the curve for \( a = 0 \) becomes a straight line, and the other curves are also modified, but they do not come significantly closer to the line for \( a = 0 \). The separation between the curves is the error in this method of approximating \( R_s(k)/\sigma^2 \), so at first glance this method does not look too promising.

However, instead of \( R_*^s (k) \), we should be considering

\[
\overline{R}_s (k) = \frac{\sin \left( \frac{\pi}{2} E_k \right) - \sin \left( \frac{\pi}{2} f(0) \right)}{\sin \left( \frac{\pi}{2} f(1) \right) - \sin \left( \frac{\pi}{2} f(0) \right)}
\]

\[
= C_1 [R_*^s (k) - C_2],
\]

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where
\[ C_2 = \sin \left( \frac{\pi}{2} f(0) \right), \]
and
\[ C_1 = (1 - C_2)^{-1}. \]
\( \overline{R}_c(k) \) is plotted as a function of \( R_c(k)/\sigma^2 \) in Fig. 2. We see that \( \overline{R}_c(k) \) is a good approximation to \( R_c(k)/\sigma^2 \) if \( R_c(k)/\sigma^2 \) is not too close to \(-1\) (which is usually true).

Forming the numbers \( \overline{R}_c(k) \) requires knowing something about the function \( f(z) \). However, they are only wanted to construct the function
\[ \overline{S}(\omega) = 1 + 2 \sum_{k=1}^{\infty} \overline{R}_c(k) \cos k\omega. \]
(12)
This function can be found more directly as
\[ \overline{S}(\omega) = C_1 [S^*(\omega) - C_2 \delta(\omega)]. \]
(13)
The presence of a dc component in \( \{y_k\} \) causes a spike in the spectrum \( S^*(\omega) \) at \( \omega = 0 \). This can be removed by inspection, which is the significance of the term \(-C_2 \delta(\omega)\) in (13). Then the multiplier \( C_1 \) can be adjusted to give whatever scale is desired for the spectrum. Thus we get \( \overline{S}(\omega) \), defined by (12), without any particular knowledge about \( a \) or the function \( f(z) \).

The error \( \overline{R}_c(k) - R_c(k)/\sigma^2 \) is negative for \( R_c(k) > 0 \), with a minimum near \( R_c(k) = 0.44\sigma^2 \) for all values of \( a/\sigma \) between 0 and 1. For \( R_c(k) < 0 \), the error is positive, increasing as \( R_c(k)/\sigma^2 \) goes from 0 to \(-1\). These errors are given at \( R_c(k)/\sigma^2 = 0.44 \) and \(-0.30 \) for several values of \( a/\sigma \) in Table 1.

Another method which we can consider for comparison is to first compute the correlation
\[ R_y(k) = \frac{E_k - f(0)}{1 - f(0)}, \]
then take
\[ \tilde{R}_c(k) = \sin \left( \frac{\pi}{2} R_y(k) \right) \]
and
\[ \overline{S}(\omega) = 1 + 2 \sum_{k=1}^{\infty} \tilde{R}_c(k) \cos k\omega. \]
This method is suggested by the original relation (3). Curves for \( \tilde{R}_c(k) \) as a function of \( R_c(k)/\sigma^2 \), similar to the curves of Fig. 2, can be plotted. These curves have the same general appearance, but they range farther from the line for \( a = 0 \). Values of the error \( \overline{R}_c(k) - R_c(k)/\sigma^2 \) are given in Table 1 for comparison. We see that this method is not as good, as well as being harder to implement.

References


Table 1. The errors in $R_z(k)$ and $R_z'(k)$ for $R_z(k)/\sigma^2 = 0.44$
(upper number) and $-0.30$ (lower number)

<table>
<thead>
<tr>
<th>$a/\sigma$</th>
<th>0.2</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_z(k) - R_z(k)/\sigma^2$</td>
<td>-0.0001</td>
<td>-0.0031</td>
<td>-0.0342</td>
</tr>
<tr>
<td></td>
<td>+0.0003</td>
<td>+0.0093</td>
<td>+0.0822</td>
</tr>
<tr>
<td>$R_z'(k) - R_z'(k)/\sigma^2$</td>
<td>-0.0031</td>
<td>-0.0192</td>
<td>-0.0738</td>
</tr>
<tr>
<td></td>
<td>+0.0064</td>
<td>+0.0378</td>
<td>+0.1261</td>
</tr>
</tbody>
</table>
Fig. 1. The function $f(z)$ for various values of $a/\sigma$

Fig. 2. $\bar{R}_s(k)$ as a function of $R_s(k)/\sigma^2$, for various values of $a/\sigma$