Symbol Stream Combining Versus Baseband Combining for Telemetry Arraying

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The objectives of this article are to investigate and analyze the problem of combining symbol streams from many Deep Space Network stations to enhance bit signal-to-noise ratio and to compare the performance of this combining technique with baseband combining. Symbol stream combining (SSC) has some advantages and some disadvantages over baseband combining (BBC). The SSC suffers almost no loss in combining the digital data and no loss due to the transmission of the digital data by microwave links between the stations. The BBC suffers 0.2 dB loss due to alignment and combining the IF signals and 0.2 dB loss due to transmission of signals by microwave links. On the other hand, the losses in the subcarrier demodulation assembly (SDA) and in the symbol synchronization assembly (SSA) for SSC are more than the losses in the SDA and SSA for BBC. It is shown that SSC outperforms BBC by about 0.35 dB (in terms of the required bit energy-to-noise spectral density for a bit error rate of $10^{-3}$) for an array of three DSN antennas, namely 64 m, 34 m(T/R) and 34 m(R).

I. Introduction

To capture signals from the Voyager spacecraft and potentially other space probes, signals from more than one antenna are combined. To reduce the cost of bringing the received signals from the Deep Space Network (DSN) and non-DSN facilities together, J.W. Layland of JPL has suggested combining quantized symbol streams in place of telemetry baseband combining.

The analysis of this article shows that symbol stream combining is superior to baseband combining. In baseband combining we require that the baseband signals from each antenna, after carrier demodulation, be brought to a signal processing center. Here the signals are delay adjusted, weighted, summed and then passed through the subcarrier demodulation assembly (SDA) and symbol synchronization assembly (SSA). The output of this processing is a quantized symbol stream which enters the maximum likelihood convolutional decoder (MCD). Since prior to decoding by the MCD all operations on the data signal are linear, it seems we might not be gaining anything by combining these signals before the SSA processing; i.e., symbols could be combined just before input to the MCD. This has the advantage of combining digital signals, which will be easier than combining baseband signals since the problems of time alignment, weighting and summing the symbol streams no longer exist. On the other hand, each station will have lower SNRs at its SDA and SSA than the SNRs from baseband combining. This article analyzes these methods of combining. It shows that these lower SNRs at the
SDA and SSA prior to combining do not present as much loss as was previously expected. On the contrary, when we compare these two methods under the same conditions, namely the same loop bandwidths for carrier tracking loop, SDA and SSA, and the same mod index, etc., we find that the overall total loss in symbol signal-to-noise ratio (SNR) for symbol stream combining is actually less than the loss in symbol SNR for baseband combining.

II. Symbol Stream Combining

A system for combining symbol streams from N stations is depicted in Fig. 1. The telemetry signal is an RF carrier that is phase-modulated by a squarewave subcarrier (sin \( \omega_{sc} t \)) at a peak modulation index \( \theta \). The subcarrier is bi-phase modulated with a binary data stream \( D(t) \). This telemetry signal is received by \( N \) ground stations. The received telemetry signal at the \( i \)th station is

\[
\eta_i(t) = \sqrt{2 P_t} \sin (\omega_c t + \Phi_{ct}) + D(t + \tau_i) \theta + \eta_i(t)
\]

where \( P_t \) is the total received power at this station, \( \omega_c \) is the carrier radian frequency, \( \Phi_{ct} \) is the carrier phase, \( \omega_{sc} \) is the subcarrier radian frequency, \( \Phi_{sc} \) is the subcarrier phase and \( \eta_i(t) \) is the additive white Gaussian noise with two-sided spectral density \( N_{0i}/2 \). The subscript \( i \) refers to the \( i \)th station throughout this article. At the output of the receiver and carrier tracking loop (CTL) the signal can be represented as

\[
\eta_i(t) = \sqrt{2 P_t} \sin \theta D(t + \tau_i) \sin (\omega_{sc} t + \Phi_{sc}) + \eta_i(t)
\]

(2)

where \( \omega_{IF} \) is IF radian frequency, \( \Phi_{ct} = \Phi_{sc} - \Phi_{sc} \) is carrier phase error and \( \Phi_{sc} \) is the phase locked loop (PLL) estimate of the carrier phase. \( \eta_i(t) \) is white Gaussian noise with two-sided spectral density \( N_{0i}/2 \). The IF carrier reference signal is

\[
r_{ct}(t) = \sqrt{2} \cos (\omega_{IF} t)
\]

(3)

The subcarrier squarewave reference signal generated by the SDA is

\[
r_{sc}(t) = \sin (\omega_{sc} t + \Phi_{sc})
\]

(4)

where \( \Phi_{sc} \) is an estimate of \( \Phi_{sc} \). After demodulating the signal \( S_i(t) \) by the reference signals in (3) and (4), we obtain

\[
W_i(t) = \sqrt{P_t} \sin \theta D(t + \tau_i) \left[ 1 - \frac{2}{\pi} |\Phi_{sc}\| \right] \cos \Phi_{ct} + \eta_i(t)
\]

(5)

which enters the SSA. The subcarrier phase error is \( \Phi_{sc} - \Phi_{sc} \) and \( \eta_i(t) \) is the baseband white Gaussian noise with two-sided spectral density \( N_{0i}/2 \).

The data waveform can be represented as

\[
D(t) = \sum_{n=-\infty}^{\infty} a_n p(t - (n-1) T_s)
\]

(6)

where \( p(t) \) is unit power rectangular pulse shape with duration \( T_s \) (symbol time) and \( a_n \) is a binary channel symbol taking on values \( \pm 1 \). Passing this signal through the SDA, assuming \( \Phi_{ct} \) and \( \Phi_{sc} \) are very slowly varying with respect to symbol time \( T_s \), we get

\[
Q_{k+m_i} = \frac{1}{T_s} \int_{(k-1)T_s+\epsilon_i}^{kT_s+\epsilon_i} \frac{\sqrt{P_t}}{\sin \theta} \left[ 1 - \frac{2}{\pi} |\Phi_{sc}\| \right] \cos \Phi_{ct} D(t + \tau_i) p(t - (k-1) T_s - \eta_i) \, dt
\]

(7)

Let \( \tau_i = m_i T_s + \epsilon_i \) where \( 0 < \epsilon_i < T_s \) for some integer \( m_i \) and \( \bar{\epsilon}_i = \epsilon_i - \epsilon_i \); then

\[
\int_{(k-1)T_s+\epsilon_i}^{kT_s+\epsilon_i} D(t + \tau_i) p(t - (k-1) T_s - \eta_i) \, dt
\]

(8)

The worst case occurs if the symbol sequence consists of alternate symbol values \( \pm 1 \), because whenever there is no
symbol transition, a time synchronization error will not affect the signal amplitude. In this case, we have

\[ Q_k \triangleq Q_{k+m_1}, t = \sqrt{P_t} \sin \theta \left[ 1 - \frac{2}{\pi} |\phi_{sc1}| \right] \cos \phi_{ct} \left[ 1 - 2 |\lambda| \right] \]

where

\[ \lambda = \frac{\tau L}{T_s} \text{ and } -\frac{1}{2} < \lambda < -\frac{1}{2} \]

Assuming the time delay for each station is perfectly estimated, then each symbol stream \( Q_{k+m_1}, t \) can be delayed by \( m_1 \) seconds.

Samples of the signal at the output of the combiner for arraying of \( N \) antennas are

\[ z_k = \sum_{i=1}^{N} \beta_i Q_{k,i} \]

where \( \beta_i \)'s are weighting factors. The optimum values of \( \beta_i \)'s will be derived shortly. Now let us find the mean and variance of \( z_k \). Given

\[ \phi_c \triangleq \{ \phi_{c1}, \phi_{c2}, \ldots, \phi_{cN} \}, \phi_{sc} = \{ \phi_{sc1}, \phi_{sc2}, \ldots, \phi_{scN} \} \]

\[ \Delta \triangleq \{ \lambda_1, \lambda_2, \ldots, \lambda_N \} \]

and \( a_k \), we have the conditional mean and the conditional variance of \( z_k \), respectively, as

\[ \bar{z}_k = \sum_{i=1}^{N} \beta_i \sqrt{P_i} \sin \theta a_k \left[ 1 - \frac{2}{\pi} |\phi_{sc1}| \right] \cos \phi_{ct} \left[ 1 - 2 |\lambda| \right] \]

and

\[ \sigma_{z_k}^2 = \frac{1}{T_s} \sum_{i=1}^{N} \beta_i^2 N_{0i} \]

Then the conditional symbol SNR, conditioned on \( \phi_c, \phi_{sc} \) and \( \Delta \), is

\[ \text{conditional symbol SNR} = \frac{\bar{z}_k^2}{\sigma_{z_k}^2} \]

and the conditional bit SNR is

\[ \text{Conditional bit SNR} = 2 \times \text{conditional symbol SNR} \]

\[ = \left[ \sum_{i=1}^{N} \beta_i \sqrt{E_{bi}} \left( 1 - \frac{2}{\pi} |\phi_{sc1}| \right) \cos \phi_{ct} \left[ 1 - 2 |\lambda| \right] \right]^2 \]

\[ \sum_{i=1}^{N} \beta_i^2 N_{0i} \]

where \( E_{bi} = 2 P_i T_s \sin^2 \theta \).

The bit error rate for \( N \) arrayed antennas, given \( \phi_c, \phi_{sc} \) and \( \Delta \), is

\[ P_b (\phi_c, \phi_{sc}, \Delta) = f \text{(conditional bit SNR)} \]

where

\[ f(x) = \begin{cases} e^{\frac{a_0 - a_1 x}{\alpha_1}} ; & x \geq \frac{a_0 + \ln 2}{\alpha_1} \\ 0.5 ; & x < \frac{a_0 + \ln 2}{\alpha_1} \end{cases} \]

\[ a_0 = 4.4314 \]

\[ a_1 = 5.7230 \]

Letting \( \beta_1 = 1 \) and optimizing \( \beta_i \)'s; \( i = 2, 3, \ldots, N \), in order to minimize the bit error rate, we get the optimum values for \( \beta_i \)'s:

\[ \beta_i^* = \sqrt{\frac{E_{bi}}{E_{b1}}} \frac{N_{01}}{N_{0i}} \]

Let

\[ \rho_i = \frac{P_i}{P_1} \cdot \frac{N_{01}}{N_{0i}} \]

and use (17) and (15); we obtain

\[ P_b (\phi_c, \phi_{sc}, \Delta) = f \left( \sum_{i=1}^{N} \rho_i N_{0i} y^2 \right) \]
where
\[ y \triangleq \sum_{i=1}^{N} \rho_i \left( 1 - \frac{2}{\pi} |\phi_{sc1}| \right) \cos \phi_{cl}(1 - 2|\lambda_i|) \]
(20)
\[ \sum_{i=1}^{N} \rho_i \]

Note that when carrier phase, subcarrier phase and symbol time are all perfectly synchronized, the value of \( y \) is one.

Assume \( \phi_{cl} \)'s, \( \phi_{sc1} \)'s and \( \lambda_i \)'s are independent of each other, having density functions \( p(\phi_{cl}) \), \( p(\phi_{sc1}) \) and \( p(\lambda) \) respectively. Then the bit error rate can be expressed as

\[
P_b = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-1/2}^{1/2} P_b(\phi_{cl}, \phi_{sc1}, \lambda) \left[ \prod_{i=1}^{N} p(\phi_{cl}) p(\phi_{sc1}) p(\lambda) \right] d\phi_{cl} d\phi_{sc1} d\lambda
\]
(21)

III. Baseband Combining

A system for arraying baseband signals out of \( N \) stations is shown in Fig. 2. Consider the baseband signals at the output of carrier tracking loops given by

\[ s_i(t) = \sqrt{2P_i} \sin \theta D(t + \tau_i) \]
\[ \xi \sin(\omega_{sc} t + \Phi_{sc}) \cos(\phi_{cl}) + n_i(t) \]
(22)

After estimating the time delays, we can align the signals in time (in practice there is about 0.2 dB loss for this alignment) and then combine them. For simplicity of analysis, assume that alignment is perfect; then at the output of the signal combiner we have

\[ S(t) = \sum_{i=1}^{N} \beta_s s_i(t + \tau_i) \]
(23)

At SDA, after demodulating \( S(t) \) by reference subcarrier squarewave reference signal,

\[ r_{sc} = \xi \sin(\omega_{sc} t + \Phi_{sc}) \]
(24)

where \( \Phi_{sc} \) is the estimate of

\[ \Phi_{sc} = \Phi_{sc} + \omega_{sc} \tau_i \]
(25)

we get

\[ W(t) = \sum_{i=1}^{N} \beta_i \sqrt{P_i} \sin \theta D(t) \left[ 1 - \frac{2}{\pi} |\phi_{sc}| \right] \cos \phi_{cl} + n_i(t) \]
(26)

entering the SSA. The \( \phi_{sc} = \Phi_{sc} - \Phi_{sc} \) is the subcarrier phase error and \( n_i(t) \) is baseband white Gaussian noise with two-sided spectral density \( N_0/2 \).

Now if we proceed similarly to the previous section, at the output of SSA we get (for alternate symbol sequence)

\[ Q_k = \sum_{i=1}^{N} \beta_i \sqrt{P_i} \sin \theta \left[ 1 - \frac{2}{\pi} |\phi_{sc}| \right] \]
\[ \cos \phi_{cl} \sin \frac{1}{2} |\lambda_i| a_k + n_k \]
(27)

where \( \lambda \) is symbol time error and \( -1/2 < \lambda < 1/2 \).

Continuing, also as in the previous section, we get

\[ P_b(\phi_{cl}, \phi_{sc}, \lambda) = f \left( \frac{\sum_{i=1}^{N} E_{bi} y}{N_{0i}} \right) \]
(28)

where

\[ y \triangleq \sum_{i=1}^{N} \rho_i \left( 1 - \frac{2}{\pi} |\phi_{sc}| \right) \cos \phi_{cl}(1 - 2|\lambda|) \]
(29)

\[ \sum_{i=1}^{N} \rho_i \]

Then the bit error rate for the BDC case is

\[ P_b = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-1/2}^{1/2} P_b(\phi_{cl}, \phi_{sc}, \lambda) \left( \prod_{i=1}^{N} p(\phi_{cl}) p(\phi_{sc}) p(\lambda) d\phi_{cl} d\phi_{sc} d\lambda \right) \]
(30)
IV. Carrier Tracking Loop Performance

The performance of carrier tracking loop (CTL) can be summarized as follows. The density function for carrier phase error can be expressed as (Ref. 1)

\[ p(\phi_c) = C \frac{\exp \left\{ \rho_{CTL} \cos \phi_c + \rho_{CTL} \phi_c \phi_{cs} \right\}}{2\pi r_0 (\rho_{CTL})} ; \quad -\pi < \phi_c < \pi \]

(31)

where \( C \) is a normalization factor such that

\[ \int_{-\pi}^{\pi} p(\phi_c)d\phi_c = 1. \]

The \( \phi_c \) is carrier phase error, \( \phi_{cs} \) is carrier static phase error and \( \rho_{CTL} \) is carrier tracking loop SNR given by (Ref. 1)

\[ \rho_{CTL} = \frac{P \cos^2 \theta}{N_0 B_L \Gamma_c} \]

(32)

In (32) \( P \) is power, and \( B_L \) is CTL loop bandwidth given by (Ref. 1)

\[ B_L = B_{L0} \left( 1 + \frac{r_0 \alpha/\alpha_0}{1 + r_0} \right) \]

(33)

where \( B_{L0} \) is loop bandwidth at threshold, \( r_0 \) is the damping parameter at threshold, and \( \alpha \) is the loop suppression factor given by (Ref. 2)

\[ \alpha = \sqrt{\frac{0.7854 \rho_{in}^2 + 0.4768 \rho_{in}^2}{1 + 1.024 \rho_{in}^2 + 0.4768 \rho_{in}^2}} \]

(34)

and \( \rho_{in} \) is the input signal-to-noise ratio to the bandpass limiter and an IF filter having bandwidth \( B_{IF} \). Defining carrier margin (CM) by (Ref. 1)

\[ CM = \frac{P \cos^2 \theta}{N_0 (2 B_{L0})} \]

(35)

then \( \rho_{in} \) can be expressed as

\[ \rho_{in} = CM \frac{2 B_{L0}}{B_{IF}} \]

(36)

Finally \( \Gamma_c \) is the limiter performance factor given by (Ref. 2)

\[ \Gamma_c = \frac{1 + \rho_{in}}{0.862 + \rho_{in}} \]

(37)

For second-order loop with transfer function

\[ F(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s} \]

(38)

where \( \tau_1 \) and \( \tau_2 \) are loop filter time constants, assume that the instantaneous Doppler offset is expressed by

\[ d(t) = \frac{\omega_c}{c} (\Omega_0 + \Lambda_0 t) \]

(39)

where \( c \) is speed of light, \( \Omega_0 \) is spacecraft speed and \( \Lambda_0 \) is spacecraft acceleration. Then the static phase error \( \phi_{cs} \) can be expressed as (Ref. 3)

\[ \phi_{cs} \approx \frac{\omega_c}{c} \left[ \frac{\Omega_0 + \Lambda_0 t + \Lambda_0 \tau_1}{G} \right] \]

(40)

where \( G \) is the loop gain

\[ G = \sqrt{\frac{K}{\pi}} \]

(41)

For a perfect second-order loop, or when \( \tau_2 << \tau_1 \), expression (40) for \( \phi_{cs} \) reduces to

\[ \phi_{cs} \approx \frac{\omega_c}{c} \frac{\Lambda_0}{\tau} \left( \frac{1 + r}{4 B_L} \right)^2 \]

(42)

Note that \( \Lambda_0 \) at Uranus is about 0.204 m/sec² and at Neptune is about 0.434 m/sec². If at threshold the desired static phase error is \( \phi_{cs0} \) then

\[ \phi_{cs} = \frac{\alpha_0}{\alpha} \phi_{cs0} \]

(43)

where \( \alpha_0 \) is suppression factor at threshold.

V. Performance of the SDA

The performance of the subcarrier demodulation assembly (Fig. 3) can be summarized as follows. The density function for subcarrier phase error can be expressed as (Ref. 1)
\[ p(\phi_{sc}) = C \frac{\exp \left( \frac{\rho_{SDA} \cos \phi_{sc} + \rho_{SDA} \phi_{sc}}{2\pi I_0 (\rho_{SDA})} \right)}{2\pi I_0 (\rho_{SDA})}; \quad -\pi \leq \phi_{sc} \leq \pi \]  

(44)

where

- \( C \) is a normalization constant as before,
- \( \phi_{sc} \) is the subcarrier phase error, \( \phi_{sa} \) is the static phase error,
- \( \rho_{SDA} \) is subcarrier loop SNR given by (Ref. 1)

\[ \rho_{SDA} = \frac{E_s}{N_0} \cdot \frac{R_s}{B_{LS}} \cdot \frac{1}{\Gamma_{SL}} \left( \frac{2\alpha'}{\pi} \right)^2 \]  

(45)

The \( E_s \) is symbol energy, \( R_s \) is symbol rate and \( \alpha' = \frac{<D(t)D'(t)>}{\text{var}D(t)} \) is a suppression factor due to data aiding. This suppression factor for various symbol transition densities is given by (Ref. 4)

Symbol transition density

0% \hspace{1cm} \alpha' = \text{erf} \left( \sqrt{\frac{2}{3} \left( \frac{E_s}{N_0} \right) e} \right) \]  

(46)

50% \hspace{1cm} \alpha' = 0.769 \left[ \frac{0.887 + 0.2 \left( \frac{E_s}{N_0} \right)^{1/2}}{1 + 0.2 \left( \frac{E_s}{N_0} \right)^{1/2}} \right] \]  

\[ \text{erf} \left( \sqrt{\frac{2}{3} \left( \frac{E_s}{N_0} \right) e} \right) \]  

(47)

100% \hspace{1cm} \alpha' = 0.538 \left[ \frac{0.687 + 0.28 \left( \frac{E_s}{N_0} \right)^{1/2}}{1 + 0.28 \left( \frac{E_s}{N_0} \right)^{1/2}} \right] \]  

\[ \text{erf} \left( \sqrt{\frac{2}{3} \left( \frac{E_s}{N_0} \right) e} \right) \]  

(48)

The soft limiter “softness” parameter which depends on the physical parameters of the limiter and the input noise power is given by (Ref. 5)

\[ D = \frac{\pi}{4} \nu^2 \text{SNR} \]  

(49)

The soft limiter factor \( \nu^2 \) is

\[ \nu^2 = \frac{1}{4} \]  

for Block III

\[ \nu^2 = \frac{1}{16} \]  

for Block IV,

and the SNR is given by

\[ \text{SNR} = \frac{E_s}{N_0} \frac{R_s}{B_{IPs}} \]  

(51)

where \( B_{IPs} \) is the bandwidth of the bandpass limiter

\[ B_{IPs} = 500 \text{ Hz} \]  

for Block III

\[ B_{IPs} = 1000 \text{ Hz} \]  

for Block IV.

Then \( \Gamma_{SL} \), the soft limiter performance factor given in Ref. 6 (Eq. 65) can be approximated (Ref. 5) by

\[ \Gamma_{SL} = \frac{1 + D}{0.862 + D} \]  

(53a)

or by (Ref. 7)

\[ \Gamma_{SL} = \frac{1 + 0.345 \alpha' \text{SNR} + 50 (\alpha' \text{SNR})^5}{0.862 + 0.690 \alpha' \text{SNR} + 50 (\alpha' \text{SNR})^5} \]  

(53b)

The loop gain of SDA is

\[ G' = \alpha' \alpha_{SL} G'_{MAX} \]  

(54)

where \( G'_{MAX} \) is the maximum gain (\( G'_{MAX} = 400 \)) and \( \alpha_{SL} \), the soft limiter average slope when the carrier tracking loop phase error is neglected, is given by (Ref. 8)

\[ \alpha_{SL} = \sqrt{\frac{D}{1 + D}} \exp \left[ - \frac{R_{SL}}{2(1 + D)} \right] \left\{ I_0 \left[ \frac{R_{SL}}{2(1 + D)} \right] \right\} + I_1 \left[ \frac{R_{SL}}{2(1 + D)} \right] \]  

(55)

for filter time constant to symbol time ratio 1/3.
where (Ref. 6)

$$R_{SL} = \left(\frac{2\alpha}{\pi}\right)^2 \text{SNR} \left(\phi_{ss}^2 + \sigma_{sc}^2\right).$$

(56)

Since $R_{SL}$ is small, $\alpha_{SL}$ can be approximated as

$$\alpha_{SL} \approx \sqrt{\frac{D}{1 + D}}$$

(57a)

or in Refs. 7 and 9 $\alpha_{SL}$ has been approximated as

$$\alpha_{SL} = \text{erf}\left[\frac{2\alpha}{\pi} \sqrt{2D}\right].$$

(57b)

$B_{LS}$ is the SDA loop bandwidth given by

$$B_{LS} = B_{LS,0} \frac{1 + \tau_0^* G'G_0'}{1 + r_0'}$$

(58)

where $B_{LS,0}$, $r_0'$, and $G_0'$ are SDA loop parameters at threshold. For the second-order loop with transfer function

$$F'(s) = \frac{1 + \tau_2^* S}{1 + \tau_1^* S}$$

(59)

as for CTL, the SDA static phase error can be expressed as

$$\phi_{ss} \approx \frac{\omega_{sc}}{c} \left[\frac{\Omega_0 + \Lambda_0 t + \Lambda_0 \tau_1^*}{t_0}'\right]$$

(60)

For a perfect second-order loop, or when $\tau_2 \ll \tau_1^*$, (60) reduces to

$$\phi_{ss} = \frac{\omega_{sc}}{c} \frac{\Lambda_0}{t_0} \left(\frac{1 + \tau_0'}{4 B_{LS}^2}\right).$$

(61)

If we set the desired static phase error at threshold as $\phi_{ss,0}$, then

$$\phi_{ss} = \frac{\omega_{sc}}{c} \frac{\Lambda_0}{t_0} \frac{\phi_{ss,0}}{G'}.$$

(62)

Finally, for SSC, the loss in data SNR due to carrier tracking loop is (Ref. 8)

$$L_{CTL,i} = \cos \phi_{ss,i} \approx \left(\frac{I_0(\rho_{CTL,i})}{I_0(\rho_{CTL,i})} \cos \phi_{ss,i}\right)^2,$$

(63)

and the loss in data SNR due to subcarrier tracking loop is (Refs. 7, 8)

$$L_{SDA,i} = \left[1 - \left(\frac{2}{\pi}\right)^{1.5} \exp\left(-\frac{\phi_{ss,0}^2}{2 \sigma_{sc}^2}\right)\right].$$

(64)

Then

$$\left(\frac{E_{sl}}{N_0}\right)_e = \frac{E_{sl}}{N_0} \times L_{SDA,i} \times L_{CTL,i}.$$

(65)

For BBC the loss in data SNR due to all carrier tracking loops is

$$L_{CTL} = \left[\sum_{i=1}^{N} \frac{I_i(\rho_{CTL,i})}{I_0(\rho_{CTL,i})} \cos \phi_{ss,i}\right]^2.$$

(66)

and loss in data SNR due to the SDA loop is

$$L_{SDA} = \left[1 - \left(\frac{2}{\pi}\right)^{1.5} \exp\left(-\frac{\phi_{ss,0}^2}{2 \sigma_{sc}^2}\right)\right].$$

(67)

Thus

$$\left(\frac{E_{sl}}{N_0}\right)_e = \left(\sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}}\right) \times L_{SDA} \times L_{CTL}.$$

(68)

VI. Performance of the SSA

The performance of the symbol synchronization assembly (Fig. 4) can be summarized as follows. The probability density function for symbol time error $\lambda$ can be expressed as

$$p(\lambda) = C \exp\left(-\frac{4\pi^2 \lambda^2 + \cos 2\pi \lambda}{(2\pi \lambda)^2}\right); \quad -\frac{1}{2} < \lambda < \frac{1}{2}.$$

(69)
where \( \lambda_\varepsilon \) is the static time shift of the loop and \( \sigma_\chi^2 \) can be expressed as (Ref. 2)

\[
\sigma_\chi^2 = \frac{\xi_0 B_{\text{LSS}}}{2 \left( \frac{E_z}{N_0} \right) e R_s} \mathcal{P}
\]  

(70)

where \( \xi_0 \) is the window size (\( \xi_0 = 1/4 \)), and the SSA loop bandwidth is

\[
B_{\text{LSS}} = B_{\text{LSS0}} \frac{1 + r''}{1 + r_0''} \sqrt{\left( \frac{E_z}{N_0} \right) e \left( \frac{E_z}{N_0} \right)_0}
\]

(71)

\((E_z/N_0)e\) is the effective input SNR to the SSA and is given by (65) for SSC and (68) for BBC. \((E_z/N_0)_0\) is the data SNR at the design point and \( r_0'' \) is the SSA loop damping parameter at the design point (\( r_0'' \approx 2 \)). \( \mathcal{P} \) is the squaring loss, given by Ref. 2,

\[
\mathcal{P} = h(0) \frac{1}{k_g^2}
\]

(72)

where

\[
h(0) = 1 + \frac{\xi_0 \left( \frac{E_z}{N_0} \right)_e - \xi_0}{2} \left[ \frac{1}{\sqrt{\pi}} e^{-\left( \frac{E_z}{N_0} \right)_e} \right] + \sqrt{\left( \frac{E_z}{N_0} \right)_e} \operatorname{erf} \left[ \sqrt{\left( \frac{E_z}{N_0} \right)_e} \right]^2
\]

(73)

and

\[
K_g = \operatorname{erf} \left( \sqrt{\left( \frac{E_z}{N_0} \right)_e} \right) - \frac{\xi_0}{2} \sqrt{\frac{1}{\pi} \left( \frac{E_z}{N_0} \right)_e} e^{-\left( \frac{E_z}{N_0} \right)_e}
\]

(74)

For a perfect second-order loop the static time error

\[
\lambda_\varepsilon = \frac{R_s}{c} \frac{\Lambda_0}{r} \left( \frac{1 + r_{ss}}{4 B_{\text{LSS}}} \right)
\]

(75)

where \( r_{ss} \) is the loop damping parameter.

The loss in data SNR due to the SSA alone is

\[
L_{\text{SSA}} = (1 - \sqrt{|\lambda|})^2
\]

(76)

\[
= \left[ 1 - 2 \sqrt{\frac{\sigma_\chi^2}{\xi_0}} e^{\frac{\lambda_\varepsilon^2}{2\sigma_\chi^2}} - 2 \lambda_\varepsilon \frac{1}{\sqrt{2\lambda_\varepsilon}} \right]^2
\]

VII. Computation of Bit Error Rate

In order to compute Eq. (21) or (30), first we examine an approximate result for the large signal-to-noise ratio case, where

\[
y \ll 1
\]

(77)

Therefore, we can make use of Taylor expansion for Eq. (19) as

\[
f \left( \sum_{i=1}^N \frac{E_{bi}}{N_{0i}} y^2 \right) = f \left( \sum_{i=1}^N \frac{E_{bi}}{N_{0i}} \right) + 2(y - 1) f' \left( \sum_{i=1}^N \frac{E_{bi}}{N_{0i}} \right)
\]

(78)

But

\[
f'(x) = \begin{cases} -\alpha_1 f(x) & x > \frac{\alpha_0 + \ln 2}{\alpha_1} \\ 0 & x < \frac{\alpha_0 + \ln 2}{\alpha_1} \end{cases}
\]

(79)

Therefore,

\[
P_b = f \left( \sum_{i=1}^N \frac{E_{bi}}{N_{0i}} y^2 \right) = \begin{cases} f \left( \sum_{i=1}^N \frac{E_{bi}}{N_{0i}} \right) \left[ 1 + 2 \alpha_1 (1 - y) \right] & \sum_{i=1}^N \frac{E_{bi}}{N_0} > T' \\ 0.5 & \sum_{i=1}^N \frac{E_{bi}}{N_0} < T' \end{cases}
\]

(80)
For SSC,

\[ \bar{y} = \sum_{i=1}^{N} \frac{\rho_i \left( 1 - \frac{2}{\pi} \left| \phi_{sc} \right| \right) \cos \phi_{ei} (1 - 2 \left| \lambda_i \right|)}{\sum_{i=1}^{N} \rho_i} \]  
(81)

For BBC,

\[ \bar{y} = \frac{\sum_{i=1}^{N} \rho_i \left( 1 - \frac{2}{\pi} \left| \phi_{sc} \right| \right) \cos \phi_{ei} (1 - 2 \left| \lambda_i \right|)}{\sum_{i=1}^{N} \rho_i} \]  
(82)

Note

\[ \frac{\cos \phi_{ei}}{\cos \phi_{ei}} \approx \frac{I_i (\rho_{CTL,i})}{I_0 (\rho_{CTL,i})} \cos \phi_{ei} \]  
(83)

for large SNR

\[ \left| \phi_{sc} \right| \approx \sqrt{\frac{2}{\pi}} e^{-2 \sigma_{sc}^{2} \phi_{sc} \sigma_{sc} + \phi_{sc} \sigma_{sc}} \text{erf} \left( \frac{\phi_{sc}}{\sqrt{2} \sigma_{sc}} \right) \]  
(84)

\[ \left| \lambda_i \right| \approx \sqrt{\frac{2}{\pi}} e^{-2 \sigma_{i}^{2} \lambda_i \sigma_{i} + \lambda_i \sigma_{i}} \text{erf} \left( \frac{\lambda_i}{\sqrt{2} \sigma_{i}} \right) \]  
(85)

where

\[ x = \sum_{i=1}^{N} \frac{E_{0i}}{N_{0i}} \gamma^2 \]  
(87)

and

\[ g(\gamma) = f(x) \]  
(88)

The \( \gamma \) is given by Eq. (20) for SSC and by Eq. (29) for BBC, and the expectation \( E \) is over all random variables-contained in \( \gamma \). Suppose we have \( M + 1 \) moments of \( \gamma \)

\[ \mu_k = E \{ \gamma^k \}; \quad k = 0, 1, 2, \ldots, M \]  
(89)

Suppose we could expand \( g(\gamma) \) as

\[ g(\gamma) \approx \sum_{i=0}^{N} \alpha_i \gamma^i \]  
(90)

Then

\[ E \{ g(\gamma) \} = \sum_{i=0}^{N} \alpha_i E \{ \gamma^i \} = \sum_{i=0}^{N} \alpha_i \mu_i \]  
(91)

Unfortunately, expansion of \( g(\cdot) \) given by (90) results in an alternating sum of moments with large coefficients \( \alpha_i \); this results in numerical inaccuracies in summing. Thus, we cannot simply expand \( g(\gamma) \) and use moments of \( \gamma \). Therefore, we would like to find the smallest number of points \( y_1, y_2, \ldots, y_p \) and weights \( w_1, w_2, \ldots, w_p \) so that the approximate discrete probability distribution

\[ \tilde{P}_r (y = y_k) = \omega_k; \quad k = 1, 2, \ldots, \nu \]  
(92)

satisfies the given moment constraints

\[ \mu_k = E(y_k) = \sum_{k=1}^{\nu} \omega_k y_k^k; \quad k = 0, 1, 2, \ldots, M \]  
(93)

More details about the following summary of the moment technique can be found in Ref. 10.

Next define the polynomial

\[ C(D) = \prod_{k=1}^{\nu} (1 - D y_k) = C_0 + C_1 D + C_2 D^2 + \cdots + C_\nu D^\nu \]  
(94)
We can show

$$
\mu_n = -\sum_{j=1}^{\nu} C_j \mu_{n-j}
$$

(95)

This form of the relationship between moments can be interpreted as a linear feedback shift register generating the moments (Fig. 5). Note that the Berlekamp-Massey linear feedback shift register synthesis algorithm can find a smallest length feedback shift register that generates \( \mu_0, \mu_1, \ldots, \mu_{\nu} \). This enables us to find \( C(D) \). Having \( C(D) \), we can find \( y_1, y_2, \ldots, y_{\nu} \). Define the polynomial

$$
P(D) = \sum_{j=1}^{\nu} \omega_j \prod_{i=1}^{\nu} (1 - D y_i) = P_0 + P_1 D + \cdots + P_{\nu-1} D^{\nu-1}
$$

(96)

Define the moment generating function polynomial as

$$
\mu(D) \triangleq \sum_{k=0}^{\nu} \mu_k D^k
$$

(97)

Then we can show

$$
p(D) = \mu(D) C(D)
$$

(98)

Finally it can be shown that

$$
\omega_k = \frac{y_k P_{\nu-1}}{C'(y_k)} ; \quad k = 1, 2, \ldots, \nu
$$

(99)

where

$$
C'(D) \triangleq \frac{d}{dD} C(D)
$$

(100)

A summary of finding \( E\{g(y)\} \) using the moment technique is shown in Fig. 6.

Define \( y \) for SSC as

$$
y \triangleq y_N = \left[ \frac{\sum_{i=1}^{\nu} \rho_i \left( 1 - \frac{2}{\pi} |\phi_{secl}| \cos \phi_{secl} (1 - 2|\lambda_i|) \right)}{\sum_{i=1}^{\nu} \rho_i} \right] \left[ \frac{\sum_{i=1}^{\nu} \rho_i}{\sum_{i=1}^{\nu} \rho_i} \right]
$$

(101)

Then

$$
y_N = y_{N-1} + \rho_N \left( 1 - \frac{2}{\pi} |\phi_{secl}| \cos \phi_{secl} (1 - 2|\lambda_N|) \right) \left( \frac{\sum_{i=1}^{\nu} \rho_i}{\sum_{i=1}^{\nu} \rho_i} \right)
$$

(102)

Finally

$$
E\{y_{N}^k\} = \sum_{i=0}^{\nu} \binom{\nu}{i} E\{y_{N}^{i}\} \rho_N^{k-i} E\left( \left( 1 - \frac{2}{\pi} |\phi_{secl}| \cos \phi_{secl} (1 - 2|\lambda_N|) \right)^{k-i} \right)
$$

(103)

Therefore, having certain moments of \( 1 - 2|\phi_{secl}| \), \( \cos \phi_{secl} \), \( 1 - 2|\lambda_N| \), by iteration using Eq. (103) we can find all required moments of \( y \). A faster method for computation of moments of \( y \) is to use the so-called semi-invariants method given in Ref. 10. Having all moments of \( y \) we can use the Berlekamp-Massey algorithm as discussed before to find required points \( y_i \) and the weights \( w_i \). Thus

$$
\cdot P_b = E\{g(y)\} = \sum_{i=1}^{\nu} \omega_i g(y_i)
$$

(104)

Example: Consider the arraying of three antennas, namely 64 m, 34 m (T/R) and 34 m (R). For a modulation index of 72%, a data symbol rate of 20 kbps (bit rate = 10 kbps), loop bandwidths at threshold of

\[
2B_{fs} (CTL) = 30 \text{ Hz}
\]

\[
B_{LS} (SDA) = 0.1 \text{ Hz}
\]

\[
B_{LSS} (SSA) = 0.05 \text{ Hz}
\]

and window size = 1/4 in SSA, we have tabulated all data SNR losses and loop SNRs for SSC and BBC. From Table 1, if we include the losses of 0.2 dB from the microwave link and of 0.2 dB due to alignment in the BBC case, then the total loss in the data SNR in SSC will be 0.35 dB less than the data SNR loss in BBC. Finally, the bit error probability curve using the moment technique is given in Figs. (7) and (8) for the above case for SSC and BBC.

**IX. Conclusion**

In this article, a system performance analysis of symbol stream and baseband combining is given. The performances of the carrier tracking loop, the subcarrier demodulation assem-
bly, and the symbol synchronization assembly have been determined. Numerical results are given for an example of arraying three antennas, namely a 64 m, 34 m (T/R) and 34 m (R) antennas of DSN. Final results show that symbol stream combining outperforms baseband combining by about 0.35 dB.

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References


Table 1. Comparison of performance of SSC and BBC

\[ \text{at } \sum_{l=1}^{3} \frac{\xi_{bl}}{N_{0l}} = 3.2 \text{ dB} \]

(all losses and loop SNR's in dB)

<table>
<thead>
<tr>
<th>Type of arraying</th>
<th>Antennas</th>
<th>Carrier tracking loop</th>
<th>SDA</th>
<th>SSA</th>
<th>Microwave loss</th>
<th>Combining loss</th>
<th>Total loss in data SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSC</td>
<td>64 m</td>
<td>-0.22 13</td>
<td>-0.044 40</td>
<td>-0.01 44</td>
<td>0</td>
<td>0</td>
<td>-0.43</td>
</tr>
<tr>
<td></td>
<td>34 m (T/R)</td>
<td>-0.56 9.2</td>
<td>-0.16 28</td>
<td>-0.05 32</td>
<td>0</td>
<td>0</td>
<td>-0.43</td>
</tr>
<tr>
<td></td>
<td>34 m (R)</td>
<td>-0.46 9.9</td>
<td>-0.12 31</td>
<td>-0.04 35</td>
<td>0</td>
<td>0</td>
<td>-0.43</td>
</tr>
<tr>
<td>BRC</td>
<td>64 m</td>
<td>-0.22 13</td>
<td>-0.03 43</td>
<td>-0.02 47</td>
<td>-0.2</td>
<td>-0.2</td>
<td>-0.78</td>
</tr>
<tr>
<td></td>
<td>34 m (T/R)</td>
<td>-0.56 9.2</td>
<td>-0.03 43</td>
<td>-0.02 47</td>
<td>-0.2</td>
<td>-0.2</td>
<td>-0.78</td>
</tr>
<tr>
<td></td>
<td>34 m (R)</td>
<td>-0.46 9.9</td>
<td>-0.03 43</td>
<td>-0.02 47</td>
<td>-0.2</td>
<td>-0.2</td>
<td>-0.78</td>
</tr>
</tbody>
</table>
Fig. 1. Block diagram of symbol stream combining system

Fig. 2. Block diagram of baseband combining system
Fig. 3. Block diagram of subcarrier demodulation assembly (SDA)

Fig. 4. Block diagram of symbol synchronization assembly (SSA)
Fig. 5. Moment generating linear feedback shift register

Fig. 6. Flow chart for computation of $E \{g(y)\}$ using moment technique
Fig. 7. Bit error rate vs total SNR for symbol stream and baseband combining for window size 1/2 (microwave and combiner losses are not included)

Fig. 8. Bit error rate vs total SNR for symbol stream and baseband combining for window size 1/4 (microwave and combiner losses are not included)