An Investigation of the Effects of Scan Separation on the Sensitivity of the SETI All Sky Survey for the Case of Gaussian Noise

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This article presents an analysis of the scalloping problem for the case of Gaussian noise statistics. We derive the optimal weighting strategy for linearly combining two observations in adjacent beam areas, and compare the sensitivity and scalloping for this weighting strategy with that realized using a single observation or using equal weighting of two observations. We also calculate the variation of the probability for detecting ETI signals with scan separation for the various weighting strategies, assuming that the transmitters are of equal strength and are uniformly distributed throughout space.

I. Introduction

One component of the Search for Extraterrestrial Intelligence (SETI) will be a survey of the entire celestial sphere over a broad frequency range to a significantly low limiting flux (Ref. 1). The inherent advantage of this strategy is that all directions are observed, and thus any signal which exceeds the threshold of the search will be located. The SETI program must design an efficient sky survey strategy which realizes the survey goals within the constraints of available antenna resources.

Time constraints and antenna dynamics dictate that the survey will be carried out by smoothly sweeping the beam across the sky. At the end of each scan, the motion must be reversed without exciting the antenna's natural modes of mechanical oscillation. At the same time the pointing in the orthogonal direction must be stepped so that the subsequent scan traverses a neighboring strip of the celestial sphere. Over a period of time, all of the celestial sphere available to the antenna will have been observed. Unless care is taken in the design of the survey, however, the sensitivity to an ETI signal will be a periodic function of position with respect to the scan pattern. This feature is commonly referred to as scalloping and its magnitude is defined to be the quantity of minimum sensitivity minus maximum sensitivity.

This article is one of a series of technical reports proceeding from the SETI sky survey definition studies and presents an analysis of the scalloping problem for the case of Gaussian noise statistics and ETI sources whose signal strength does not change with time. We derive an optimal weighting strategy for linearly combining data in two adjacent beam areas and then compare the sensitivity and scalloping achieved using this strategy with that realized utilizing practical weighting strategies. Finally, we derive the variation with scan separation of the probability for detecting ETI signals, assuming that the transmitters are of equal strength and uniformly distributed throughout space. Subsequent papers will extend our analyses to the case of non-Gaussian noise statistics, the realm in which the contemplated sky survey will be operating due to the small number of independent samples comprising an accumulation.

II. Derivation of the Optimal Weighting Strategy

In this section, we derive an optimal strategy for linearly combining two observations to achieve the maximum sensitivity to a signal. To simplify the calculation, we assume a one dimensional model (see Fig. 1). The x-axis represents the loci of possible source positions between scan tracks, and we search by stepping a symmetrical beam, $f(x)$, in increments of x_0 between the scans. The y-axis represents the gain of the beam, and the scan direction is normal to the plane of the figure. Suppose that two neighboring beams straddle the source position, x_s, and that the first beam is located on the origin of the x-axis and is the nearer of the two to x_s. Thus the gain of the first beam applied to the signal, f_1, is larger than that of the second beam, f_2.

The criterion for detection of the signal is an excess noise power observed in the receiver attached to the one dimensional antenna. If the beam is located far from the source position, the noise power distribution function is Gaussian with an expectation, $\langle n \rangle$, and a variance, σ^2. If the beam is near the source position, the noise power distribution function is different in that its expectation is augmented by the product of the source strength and beam gain (see Fig. 2). We wish to design an efficient algorithm to detect this signal consistent with a previously set probability that it is caused by noise alone, α_0, and probability of missing the signal, β_0.

In the discussion which follows we shall employ the notation:

f_1 = the gain of the (nearer) first beam area at x_s

f_2 = the gain of the second beam area at x_s

n_1, n_2 = power received due to background noise in each of the two beam areas

s = time invariant strength of the source

p_1 = received power from first beam = $n_1 + f_1 s$

p_2 = received power from second beam = $n_2 + f_2 s$

Th = power threshold which must be exceeded to satisfy detection criterion

α_0 = probability of false alarm

β_0 = probability of missing signal

$Pr\{A|B\}$ = probability of A gvien that B is true

Consider two detection algorithms:

A1. Treat each beam area independently, choosing a threshold Th_1 such that:

$$\alpha_0 \geqslant Pr\ \{p_1 \geqslant Th_1 | \text{noise alone}\} \tag{1}$$

$$\beta_0 \geqslant Pr\ \{p_1 \leqslant Th_1 | \text{noise + signal } s_1\} \tag{2}$$

Since we are assuming that $f_1 \geqslant f_2$, the weakest source which may be detected is one for which $p_1 \geqslant Th_1$. Let s' be the strength of this weakest source, thus:

$$s_1 = f_1 s' \tag{3}$$

A2. Linearly combine the powers observed in the two beams by means of a weighting function, w, and then apply a threshold Th_2. Assume the same α_0 and β_0 as in algorithm A1:

$$p_3 = wp_1 + (1 - w)p_2 \tag{4}$$

$$\alpha_0 \geqslant Pr\ \{p_3 \geqslant Th_2 | \text{noise alone}\} \tag{5}$$

$$\beta_0 \geqslant Pr\ \{p_3 \leqslant Th_2 | \text{noise + signal } s_3\} \tag{6}$$

Thus a source is detected if $p_3 \geqslant Th_2$. Let s'' be the strength of the weakest source that may be detected by this algorithm, thus:

$$s_3 = wf_1 s'' + (1 - w)f_2 s'' \tag{7}$$

The analysis which follows derives the ratio of the sensitivities of these two detection algorithms. Then the weight, w, is optimized so that s'/s'' is a maximum. Since the noise is Gaussian, p_1 and p_2 are also Gaussian with probability distribution functions:

$$P(p_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(p_i - \langle n \rangle)^2}{2\sigma^2}\right] \tag{8}$$

if no signal is present; and

$$P(p_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(p_i - \langle n \rangle - s_i)^2}{2\sigma^2}\right] \tag{9}$$

if signal s_i is present (where, i = 1,2). Similarly,

$$P(p_3) = \frac{1}{\sqrt{2\pi}\,\sigma_3} \exp\left[-\frac{(p_3 - \langle n \rangle)^2}{2\sigma_3^2}\right] \tag{10}$$

if no signal is present; and if signal s_3 is present,

$$P(p_3) = \frac{1}{\sqrt{2\pi}\,\sigma_3} \exp\left[-\frac{(p_3 - \langle n \rangle - s_3)^2}{2\sigma_3^2} \right] \quad (11)$$

where we have defined:

$$\sigma_3^2 = [w^2 + (1-w)^2]\,\sigma^2 \quad (12)$$

We are now in a position to derive an expression for the ratio of the limiting sensitivities of the two detection algorithms, s'/s'', which can then be maximized to find the optimal weight. The notation for the derivation is simplified if we define $\Phi_o(z)$:

$$\Phi_o(z) \equiv \frac{1}{\sqrt{2\pi}} \int_z^\infty \exp - \frac{t^2}{2}\, dt \qquad \text{for } z \geq 0$$

Assuming that $f_1 > f_2$, for strategy A1 we see from Eqs. (1) and (8) (the noise only case):

$$\alpha_0 = Pr\,\{p_1 \geq Th_1 \,|\, s_1 = 0\} = \Phi_o\left(\frac{Th_1 - \langle n \rangle}{\sigma}\right) \quad (13)$$

and from Eqs. (2) and (9) (the signal present case):

$$\beta_0 = Pr\,\{p_1 \leq Th_1 \,|\, s_1 > 0\} = \Phi_o\left(\frac{s_1 + \langle n \rangle - Th_1}{\sigma}\right) \quad (14)$$

On the other hand, for strategy A2 we see from Eqs. (5) and (10) (the noise only case):

$$\alpha_0 = Pr\,\{p_3 \geq Th_1 \,|\, s_3 = 0\} = \Phi_o\left(\frac{Th_2 - \langle n \rangle}{\sigma_3}\right) \quad (15)$$

and from Eqs. (6) and (11) (the signal present case):

$$\beta_0 = Pr\,\{p_3 \leq Th_2 \,|\, s_3 > 0\} = \Phi_o\left(\frac{s_3 + \langle n \rangle - Th_2}{\sigma_3}\right) \quad (16)$$

Now recall that we required α_0 to be the same for both algorithms. Thus Eqs. (13) and (15) are equal:

$$\left(\frac{Th_1 - \langle n \rangle}{\sigma}\right) = \left(\frac{Th_2 - \langle n \rangle}{\sigma_3}\right) \quad (17)$$

Similarly, we require β_0 to be the same for both algorithms. Thus Eqs. (14) and (16) are also equal:

$$\left(\frac{s_1 + \langle n \rangle - Th_1}{\sigma}\right) = \left(\frac{s_3 + \langle n \rangle - Th_2}{\sigma_3}\right) \quad (18)$$

Combining Eqs. (17) and (18), we have $s_1/s_3 = \sigma/\sigma_3$. We may use this result and Eqs. (3), (7), and (12) to find the ratio of the sensitivities of the two detection algorithms:

$$\frac{s'}{s''} = F(w) = \frac{w + \frac{f_2}{f_1}(1-w)}{\sqrt{w^2 + (1-w)^2}} \quad (19)$$

We derive the optimal weighting strategy for combining the two responses so that the ratio of the sensitivities for the two detection algorithms is maximized by solving $dF(w)/dw = 0$:

$$w_{opt} = \frac{f_1}{(f_1 + f_2)} \quad (20)$$

Since we have no foreknowledge of the position of the ETI source, we must employ our assumption that the signal strength is time invariant and rewrite Eq. (20) in terms of the received power and instantaneous noise power:

$$w_{opt} = \frac{(p_1 - n_1)}{(p_1 - n_1) + (p_2 - n_2)} \quad (21)$$

The noise powers contained within the observed powers are not available to the observer and can only be estimated by calculating the expectation value. A subsequent paper (Lokshin, in preparation) will show that the optimal weighting strategy cannot be used to increase the signal to noise ratio due to the presence of cross products between the error in the estimation of the weights and the difference between the two received powers. The optimal weighting calculation does, however, give us a theoretical limit which we may use to judge practical weighting strategies.

III. Some General Results

We shall compare the theoretical limit to the enhancement of sensitivity and minimization of scalloping realized by the optimal weighting strategy to the sensitivity and scalloping (1) achieved by utilizing a single observation and (2) achieved by combining two observations with equal weights.

Suppose the sky survey could detect an ETI source of strength, s_0, if the center of the beam passes directly over the source position. What would be the sensitivity if two adjacent scans bracket the source position? Obviously, if information from only one beam area is used (strategy A1) the detectable signal is a function of the distance the source is offset from the nearer beam center (beam number 1 is assumed to be nearer):

$$s'(x_s) = s_0/f(x_s) = s_0/f_1 \tag{22}$$

Substituting Eq. (22) into (19) we find the expression for the detectable signal if one or two beam areas are used with arbitrary weighting:

$$s''(x_s) = \frac{s_0}{f(x_s) \cdot F(w)} = \frac{s_0\sqrt{w^2 + (1-w)^2}}{wf_1 + (1-w)f_2} \tag{23}$$

This expression can now be employed to compare the peak sensitivity and scalloping as a function of x_0 for arbitrary weighting strategies, wherein it is assumed that the sky survey is carried out by scanning along parallel tracks which are separated by x_0.

Note that if $w = 1$, Eq. (23) reverts to Eq. (22), the single beam area case. This choice leaves the survey exposed to the full effect of scalloping inherent in the interscan separation. On the other hand, if we combine information from two beam areas in adjacent scans, the variation in detectable signal strength with x_s can be decreased. In the case in which the two beam areas are weighted equally without giving attention to the strength of the signal in each we find:

$$s''(x_s) = \frac{s_0\sqrt{2}}{f_1 + f_2} \tag{24}$$

This choice of weighting function minimizes the scalloping in sensitivity at the price of degrading the peak sensitivity. Of course, application of the optimal weighting strategy achieves

$$s''(x_s) = \frac{s_0}{\sqrt{f_1^2 + f_2^2}} \tag{25}$$

Given the symmetrical beam shape, $f(x)$, these expressions may be evaluated for all x_s to determine the relative sensitivities of the three strategies and their scalloping as a function of x_0. In fact, we have done so for the case in which the beam shape is a Gaussian of arbitrary half power beam width (HPBW):

$$f(x) = \exp\left[-4\ln(2)\left(\frac{x}{\mathrm{HPBW}}\right)^2\right] \tag{26}$$

Figure 3 shows the relative sensitivities as a function of x_s of the three strategies for a scan separation, $x_0 = 1$ HPBW. The x-axis is shown only over the range $0 \leqslant x_s \leqslant x_0$ since the response is symmetrical about either limit. For the case in which data from only a single beam area is used, scalloping is 3 dB. For the case in which data from two neighboring beam areas are combined with equal weight, scalloping is reduced to 0.34 dB. This is achieved at the cost of a loss in peak sensitivity of 1.17 dB relative to the single beam result due to averaging data containing almost no signal with data having the maximum signal to noise ratio. For the case in which the optimal weight has been used, scalloping is reduced to 1.5 dB while peak sensitivity is enhanced 0.01 dB relative to the single beam result.

Figure 4 shows the relative sensitivities as a function of x_s for a scan separation, $x_0 = 0.75$ HPBW. Scalloping for the single beam case is reduced to 1.7 dB whereas scalloping for the equal weight case is increased to 0.48 dB and the loss in peak sensitivity relative to the single beam result is decreased to 0.2 dB. Note that the larger scalloping is due to a greater enhancement of sensitivity between the beam areas than at their centers. This effect was just barely noticable in the preceeding figure. Scalloping for the optimal weight case is reduced to 0.3 dB and the peak sensitivity is enhanced by 0.1 dB relative to the single beam result.

Figure 5 shows the variation of scalloping as a function of x_0 for the three weighting strategies. For large x_0 the improvement in scalloping over that resulting from a single beam is 3 dB for equal weighting and 1.5 dB for optimal weighting. Note, however, that minima occur in the scalloping for the equal weighting and optimal weighting cases. For a scan separation smaller than 0.95 HPBW, the sensitivity to a source located between equally weighted scans increases faster than the sensitivity to a source located at the center of either scan as the scan separation is decreased. Thus the nonuniformity in sensitivity increases with decreasing scan separation until a substantial overlap is achieved. A similar phenomenon occurs for the optimally weighted scans after x_0 shrinks below 0.65 HPBW.

Figure 6 shows the variation of the peak sensitivity as a function of x_0 relative to that achieved using only a single beam. The optimal weighting strategy always achieves a better sensitivity than a single beam, albeit the improvement is not very great for $x_0 \geqslant 0.65$ HPBW. The equal weighting strategy peak sensitivity has already dropped halfway to its asymptotic minimum at $x_0 \approx 0.85$ HPBW.

The variations in scalloping and peak sensitivity shown in Figs. 5 and 6 are key considerations in the design of a sky survey strategy and will impact the survey sensitivity, given the constraints on antenna time and available memory and processing power. The sensitivity of the survey can be increased by (1) dwelling longer in each beam area, (2) decreasing the scan separation, (3) combining data from neighboring beam areas, or (4) combining all three options. Each option has its price, however.

In the first option, the time to complete the survey increases as the square of the ratio of new to old sensitivities, but the scalloping is not affected. In the second option, the time increases directly as the ratio of old to new scan separations increases and the scalloping is reduced. In the third option, survey time is not affected and scalloping is reduced, but memory and processing requirements increase and peak sensitivity may suffer depending upon the manner of combination chosen. The fourth option will always be chosen, but the mix will vary depending upon the constraints under which the survey will operate.

IV. The Effect of Scan Separation in the Case of Uniformly Distributed Transmitters and Fixed Survey Time

Plausible arguments may be advanced for any number of assumptions about the spatial, power, duty cycle, and transmitted frequency distribution functions for signals of ETI origin. Given a set of assumptions, a survey may be tailored to maximize the probability that it will detect a signal of that class. Many reasonable scenarios have been advanced in the literature, but the great advantage of an all sky survey lies in the fact that it incorporates the fewest *a priori* assumptions.

We now consider the impact of scan separation and weighting strategy upon the probability of detecting an ETI signal. The calculation requires that some assumptions be made about the distribution of sources and the manner in which the survey will be carried out. In the light of real life constraints for a sky survey and our state of ignorance concerning possible ETI sources, we shall follow in the footsteps of Drake (Ref. 2) and Gulkis (Ref. 3) and assume that:

(a) An $M \times N$ HPBW^2 area of the sky is to be surveyed in a fixed time, T.

(b) The transmitters are of equal strength and are distributed uniformly throughout space.

Suppose that our hypothetical sources each have an effective isotropic radiation power, P, and the boresight gain of the antenna is G. If a particular source is a distance, R, from earth and is displaced relative to the boresight by a distance, x, the flux seen by the receiving system in one beam area which is due to the source is

$$s(x) = \frac{G \cdot P}{4\pi R^2} \cdot f(x) \tag{27}$$

If the source is to be detected, this flux must be greater than or equal to the minimum detectable flux, s'. Thus, the source will be detected if it is closer than the distance R_m:

$$R_m(x) = \left[\frac{G \cdot P}{4\pi s'} \cdot f(x)\right]^{1/2} \tag{28}$$

Equation (19) may be substituted into (28) to find the expression for the maximum distance if the weighted data from two beam areas are used. We must keep in mind that $f(x)$ is really f_1 in Eq. (19), and we must explicitly show the dependence upon the integration time, τ, since $s_0 \sim \tau^{-1/2}$:

$$R_m(x) \sim \left[\frac{G \cdot P}{s''} \cdot f(x) \cdot F(w) \cdot \sqrt{\tau}\right]^{1/2} \tag{29}$$

The number of detectable transmitters (and thus the probability of detecting one) is proportional to the volume of space observed:

$$\delta Pr\{\text{detection}\} \sim \delta V = R_m^3 \, \delta\Omega \tag{30}$$

Substituting Eq. (29) into (30), we have:

$$\delta Pr\{\text{detection}\} \sim \left[\frac{GP}{s''}\right]^{3/2} \tau^{3/4} \, \{f(x) \cdot F(w)\}^{3/2} \, \delta\Omega \tag{31}$$

Assumption (a) may now be applied as a constraint so that the effect of the scan separation, x_0, upon the probability of detecting a signal may be evaluated. The simplest survey strategy entails making scans which are N HPBWs long, stepping by x_0 in the orthogonal coordinate until a distance of M HPBWs is covered. Thus the number of scans is equal to M/x_0. The integration time is set equal to some fraction of the amount of time required to scan through one HPBW, and we shall assume here that the fraction is unity. Thus the total time allowed for the survey (assuming zero time between scans) must be:

$$T = \frac{M}{x_0} \cdot N \cdot \tau \tag{32}$$

Solving for τ, we may now integrate Eq. (31) over all x_s to derive an expression for the probability of detecting a signal as a function of scan separation:

$$Pr\{\det | x_0\} \sim \left[\frac{MN}{x_0}\right]^{1/4} \left[\frac{GP\sqrt{T}}{s''}\right]^{3/2} \times 2\int_0^{1/2 x_0} \left[\frac{wf(x) + (1-w)f(x_0 - x)}{\sqrt{w^2 + (1-w)^2}}\right]^{3/2} dx \tag{33}$$

Figure 7 shows the variation of a normalized $Pr\{\text{detection}|x_0\}$ with x_0 for the three weighting strategies. The normalization is chosen so that the maximum relative probability for the optimal weighting strategy is unity. As x_0 becomes larger, the scans can be slowed down and still allow the survey to cover the same area of sky in the given time limit. However, the scalloping in sensitivity increases with x_0 and degrades the probability of detection. For very small values of x_0, the optimal weighting and equal weighting strategy improve the probability over that achieved by the single beam strategy by the expected ratio of $2^{3/4}$ due to the effective doubling of integration time on source. As the scan separation increases the single beam area strategy approaches the optimal result and the equal weighting strategy falls off to the expected ratio of 1/2 due to doubling of the noise.

The relative probabilities of detection for the single beam strategy and the equal weighting strategy are equal for a scan separation of about 0.8 HPBW, and are degraded relative to the optimal result by approximately 15%. The probability of detection achieved by the equal weighting strategy peaks at a scan separation of 0.6 HPBW, and it is degraded from the optimal peak by 8%. The probability of detection achieved by using a single beam area peaks at a scan separation of 1.3 HPBW, and it is degraded from the optimal peak by 5%. It is clear that the scalloping allowable in a survey will depend upon the assumptions of the designers of the search strategy.

V. Suggestions for Further Analysis

It is possible to extend the foregoing analysis to cover the general case of combining N beam areas. Of more immediate concern, however, is an extension to the case of a non-Gaussian noise statistic. The contemplated high speed all sky survey will operate in this domain due to the small number $(4 < n < 100)$ of independent samples which will be combined before thresholding. A series of papers is in preparation which will cover this topic.

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References

1. Gulkis, S., Olsen, E. T., and Tarter, J., "A Bimodal Search Strategy for SETI," *Strategies for the Search for Life in the Universe*, edited by M. D. Papagiannis. The Proceedings of the IAU General Assembly, Vol. 83, pp. 93-105. D. Reidel, Dodrecht, Holland, the Netherlands, 1980.

2. Drake, F. D., "Estimates of the Relative Probability of Success of the SETI Search Program," *SETI Science Working Group Report*, edited by F. Drake, J. H. Wolfe, C. L. Seeger. NASA Technical Report, p. 67, October 1983.

3. Gulkis, S. "Note on the Optimal Search Strategy for Uniformly Distributed CW Transmitters," *TDA Progress Report 42-77*, January-March 1984, Jet Propulsion Laboratory, Pasadena, California.

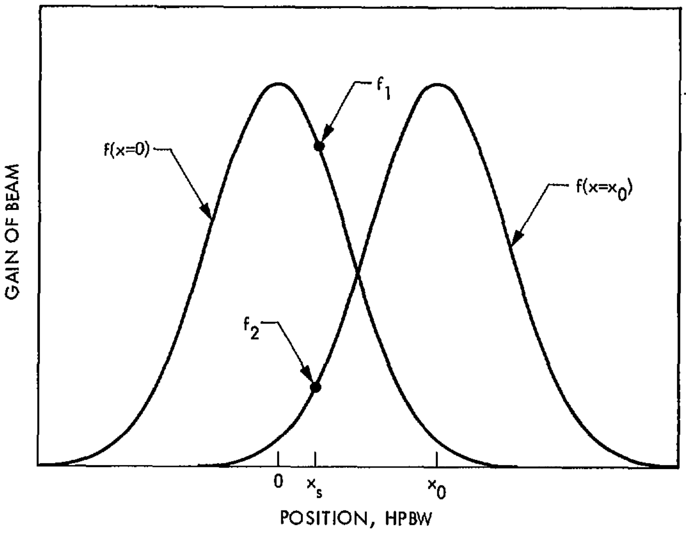

Fig. 1. One dimensional model of survey geometry

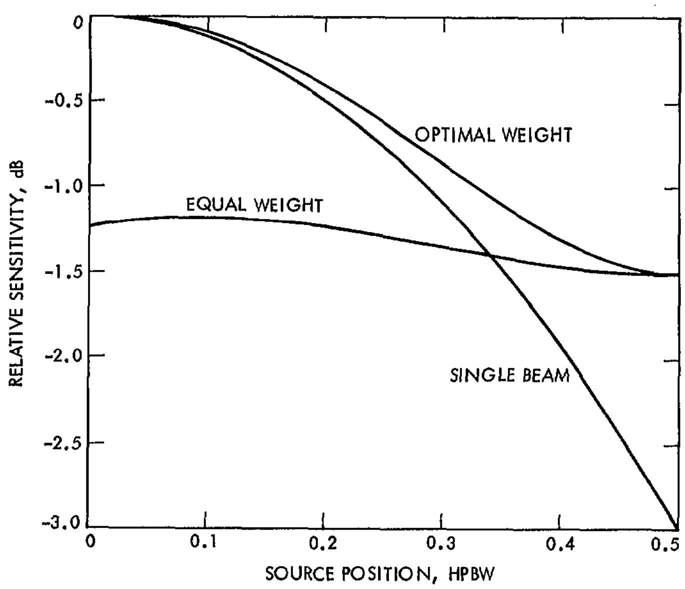

Fig. 3. Relative sensitivity as a function of source position for a scan separation of 1.0 HPBW

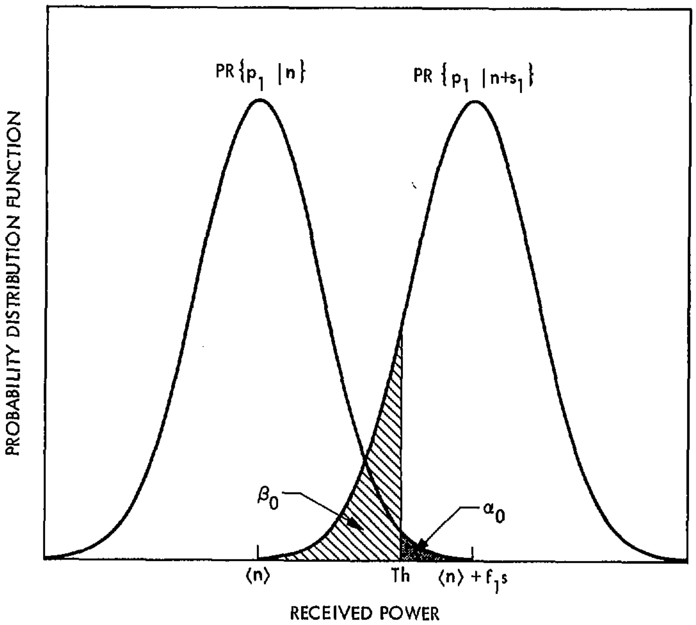

Fig. 2. Probability distribution functions of received power due to noise alone and due to noise and source

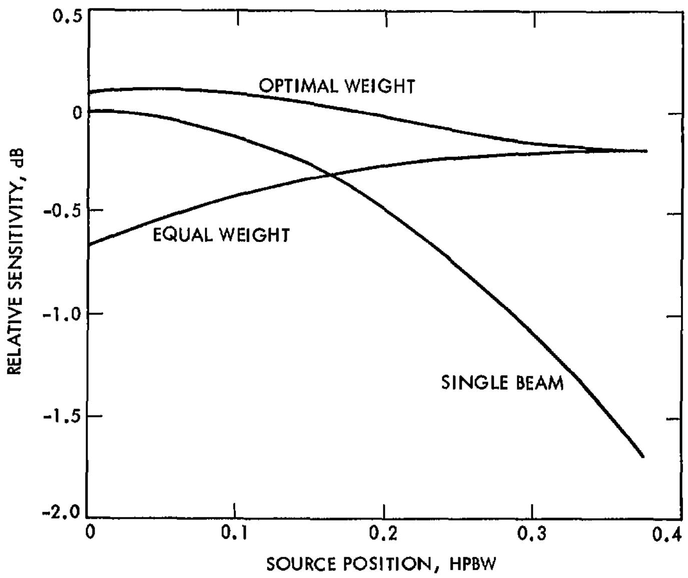

Fig. 4. Relative sensitivity as a function of source position for a scan separation of 0.75 HPBW

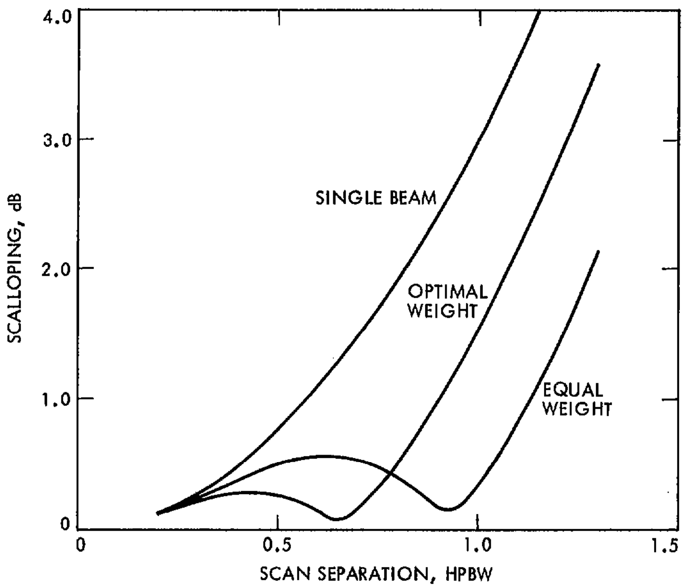

Fig. 5. Scalloping as a function of scan separation

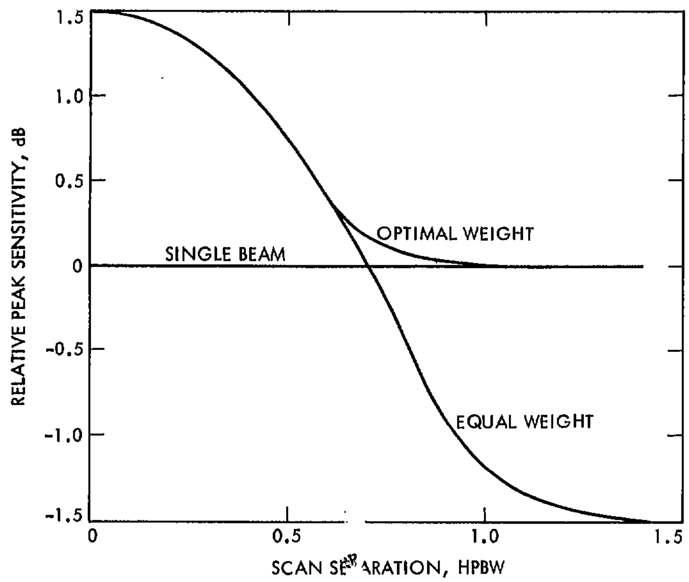

Fig. 6. Relative peak sensitivity as a function of scan separation

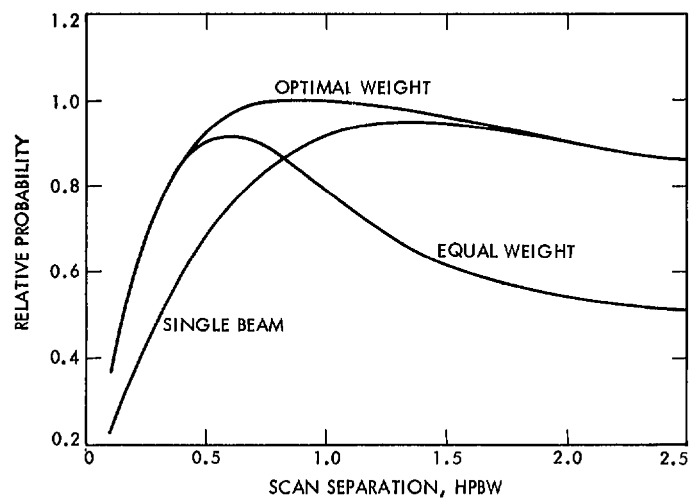

Fig. 7. Normalized relative probability to detect ETI as a function of scan separation

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