Digital Filters for Digital Phase-Locked Loops

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An s/z hybrid model for a general phase-locked loop is proposed in this article. The impact of the loop filter on the stability, gain margin, noise-equivalent bandwidth, steady-state error and time response is investigated. A specific digital filter is selected which maximizes the overall gain margin of the loop. This filter can have any desired number of integrators. Three integrators are sufficient in order to track a phase jerk with zero steady-state error at loop update instants. This filter has one zero near \( z = 1.0 \) for each integrator. The total number of poles of the filter is equal to the number of integrators plus two.

I. Introduction

In this article, the impact of a general digital filter on the gain margin, noise-bandwidth, steady-state error and transient response of a digital phase-locked loop is investigated. For the proposed s/z-domain model of a digital phase-locked loop, a wide variety of digital filters was investigated by analysis and by simulation.

A specific type of filter is suggested which has the property of having one zero for each filter integrator. The total number of poles of this filter equals the number of integrators (poles at \( z = 1 \)) plus two. All loop poles are forced to remain on or near the real axis for the maximum possible range of loop gain. The loop remains stable with any number of integrators. The noise bandwidth increases slightly with the number of integrators. Gain margin and transient response are almost insensitive to the number of integrators.

The proposed type of filter allows tracking very high Doppler rates with zero steady-state error (at loop update instants).

II. Loop Components

The basic components of our digital loop (see Fig. 1) are:
(a) an integrate-and-dump circuit, which also serves as an analog-to-digital converter, (b) a digitally controlled oscillator (DCO), which also serves as a digital-to-analog converter, and (c) a loop filter. These components are modeled in the hybrid s/z domain as follows:

(a) Integrate-and-dump (According to Ref. 1):

\[
I(s, z) = \frac{e(z)}{\phi(z)} = G \frac{1}{s} \frac{z - 1}{z} \quad (1)
\]
(b) DCO:\[
D(s, z) = \frac{\hat{\phi}(s)}{\hat{\theta}_d(z)} = \frac{1 - \exp(-sT)}{s^2} = \frac{1 - z^{-1}}{z^2} \quad (2)
\]

(c) Loop Filter:
\[
F(z) = \frac{\hat{\theta}(z)}{\epsilon(z)} = K_f \frac{(1 - z_1 z^{-1}) (1 - z_2 z^{-1}) \ldots (1 - z_m z^{-1})}{(1 - p_1 z^{-1}) (1 - p_2 z^{-1}) \ldots (1 - p_n z^{-1})} \quad (3)
\]

where

- \( G_Q \) represents the combined gain of the phase detectors, and other loop components and is proportional to the power of the input signal.
- \( T \) is the loop update time
- \( \{z_i\} \) are the zeros of the loop filter
- \( \{p_i\} \) are the poles of the loop filter

In the actual implementation of the loop filter, there is a time delay between the instants the error signal \( e_i \) is read and the estimated phase rate \( \hat{\theta}_i \) is computed. This time delay is modeled as
\[
T_d(s) = \frac{\hat{\theta}_d(z)}{\hat{\theta}(z)} = \exp(-s g T_d) \quad (4)
\]

where \( 0 < g < 1.0 \) is the normalized time delay

\[
g = \frac{T_c}{T} \quad (5)
\]

The term \( T_c \) is the computation time, and \( \hat{\theta}_d(z) \) is \( \hat{\theta}(z) \) delayed by \( T_c \) seconds.

Referring to Fig. 1, we see that
\[
X(s) = \frac{\phi(s)}{s} = \frac{\theta(s)}{s} - \hat{\theta}(z) [1 - \exp(-sT)] \frac{\exp(-s g T_d)}{s^3} \quad (6)
\]

Taking the \( z \) transform of Eq. (6), we obtain\(^2\)
\[
X(z) = \left( \frac{\theta(s)}{s} \right)^* = \left( \frac{\theta(s)}{s} \right)^* - \frac{T^2}{2} \left[ \frac{(1 - g^2) z^2 + (1 + 2g - 2g^2) z + g^2}{z(z - 1)^2} \right] \hat{\theta}(z) \quad (7)
\]

where the asterisk denotes the \( z \) transform of the expression inside the parentheses. Equation (7) has the block diagram representation of Fig. 2. Using the above equation, together with Eqs. (1) and (3), the open-loop transfer function is
\[
G(z) \triangleq \frac{\left( \frac{\hat{\theta}(s)}{s} \right)^*}{\frac{\hat{\theta}(s)}{s}} \cdot \frac{1}{X(z)} = \frac{\hat{\theta}_1(z)}{X(z)}
\]

\[
= G \frac{(z^2 + C_1 z + C_2)}{z^2(z - 1)} F(z) \quad (8)
\]

where

\[
G = \frac{G_Q T^2}{2} (1 - g)^2 \quad (9)
\]

is the effective loop gain

\[
C_1 = \frac{(1 + 2g - 2g^2)(1 - g^2)}{1 - g^2} \quad (10)
\]

\[
C_2 = g^2 / (1 - g)^2 \quad (11)
\]

Using Eq. (8), the closed-loop transfer function will be
\[
H(z) \triangleq \frac{\hat{\theta}_1(z)}{(\hat{\theta}(s)/s)^*} = \frac{G(z)}{1 + G(z)} \quad (12)
\]

### III. Selection of the Loop Filter

In general, a digital filter can be expressed by the following difference equation:
\[
\hat{\theta}_i = \sum_{j=1}^{n} a_j \hat{\theta}_{i-j} + \sum_{j=0}^{m} b_j e_{i-j} \quad (13)
\]

\(^{1}\)The DCO can be modeled as a Digital to Analog converter for which the transfer function is
\[
\frac{\hat{\theta}(s)}{\hat{\theta}_d(z)} = \frac{1 - \exp(-sT)}{s} \quad (2)
\]

Multiplying this by \( 1/s \) converts \( \hat{\theta}(s) \) to \( \hat{\theta}(s) \).

\(^{2}\)In general, given a function \( L(s) \), we always have:
\[
[\exp(-s g T) L(s)]^* \triangleq L(z, m), \quad m = 1 - g
\]

where \( L(z, m) \) is the modified \( z \) transform of \( L(s) \). This technique is used to obtain Eq. (7) from Eq. (6).
The direct form I realization of this filter is shown in Fig. 3. Figure 4 depicts the same filter in the direct form II realization. The transfer function corresponding to Eq. (13) is

\[ F(z) = \frac{\beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \ldots + \beta_m z^{-m}}{1 - \alpha_1 z^{-1} - \alpha_2 z^{-2} - \ldots - \alpha_n z^{-n}} \]  

(14)

which can be factored into the pole-zero representation of Eq. (3).

With \( \beta_0 = 1 \) and \( m < n \), Eq. (14) can be rewritten as follows:

\[ F(z) = \frac{(z^m + \beta_1 z^{m-1} + \ldots + \beta_m) z^{n-m}}{z^n - \alpha_1 z^{n-1} - \ldots - \alpha_n} \]  

(15)

\[ = \frac{(z - z_1) (z - z_2) \ldots (z - z_m) z^{n-m}}{(z - p_1) (z - p_2) \ldots (z - p_n)} \]  

(16)

This filter will always have \( n \) poles and \( n \) zeros, such that \( n - m \) of the zeros will be at \( z = 0 \).

The filter controls the following four main parameters of the phase-locked loop:

1. Gain margin
2. Noise-equivalent bandwidth
3. Steady-state error
4. Transient response

The selection of proper values for \( z_i \) and \( p_i \) is dictated by the necessity of optimizing the above loop parameters. In what follows, we will address each optimization criterion individually.

**A. Stability and Gain Margin**

Inserting Eq. (16) in Eq. (8) and using Eq. (12), the closed loop transfer function with an \( n \)th order filter is

\[ H(z) = \frac{G(z - z_a) (z - z_b) (z - z_1) (z - z_2) \ldots (z - z_m) z^{n-m}}{z^m (z - 1) (z - p_1) (z - p_2) \ldots (z - p_n)} \]  

(17)

Here \( z_a \) and \( z_b \) are the roots of \( (z^2 + C_1 z + C_2) \) in Eq. (8). These two zeros result from nonzero computation time. For example, assuming that \( g = 0.5 \) (a computation time of half an update time), then \( z_a = -0.7176 \) and \( z_b = -5.8283 \).

For the effective loop gain \( G \) changing from \( G = 0 \) to \( G = \infty \), the poles of \( H(z) \) will move starting at the location of the open-loop poles, \( p_n \), of \( G(z) \) and ending at the location of the open-loop zeros, \( z_p \), of \( G(z) \). The loop will be stable when all the poles of \( H(z) \) are inside the unit circle in the complex \( z \)-plane. Thus, in order to increase the gain margin, given that other constraints are met, we want to place all \( z_i \)'s and \( p_i \)'s such that the range of \( G \) that maintains a stable loop is maximum.

In order to track high Doppler rates with zero or minimum steady-state error, the loop should have 2 or more integrators, i.e., open-loop poles at \( z = 1.0 \). For stability, we want these poles to move inside the unit circle with increasing \( G \). The closer the zeros are to the point \( z = 1.0 \), the faster the poles at \( z = 1.0 \) will move inside the unit circle. Since here all gains and loop parameters are controlled digitally, there is no danger of instabilities due to drifts of gain values as is the case with analog loops. However, it should be checked that any quantization and truncation errors will not produce undesirable fluctuations in the locations of \( z_i \) and \( p_i \).

Performing a root-locus analysis and simulation revealed that the range of allowable \( G \) values increases when the poles are forced to stay longer on the real axis. A rule of control theory states that the root locus on the real axis always lies in a section of this axis to the left of an odd number of poles and zeros. Figure 5 illustrates some root locus diagrams with the number of integrators, \( N \), in \( F(z) \) being a parameter.

Using this simple rule, the minimum number of zeros near \( z = 1.0 \) was determined. For each pole of \( F(z) \) at \( z = 1.0 \), one zero is required. These \( m = N \) zeros will force the poles at \( z = 1.0 \) to move inside the unit circle as shown in Fig. 5. Note from Eq. (8) that the system transfer function contributes one pole at \( z = 1.0 \). Thus, the total number of poles at \( z = 1.0 \) (integrators) is \( N + 1 \).

By trial and error, it was found that, with any number of integrators, the gain margin increases when the open loop transfer function, \( G(z) \), has a minimum number of poles. Since the transfer function contributes two poles at \( z = 0 \) (see Eq. [8]), which try to move quite fast out of the unit circle when the loop gain is increased, it is desirable to cancel these two poles with two zeros at the origin. From Eq. (16), this implies that \( n - m = 2 \). Since, from the above, the number \( m \) of zeros not at the origin was selected equal to the number \( N \) of integrators, then the total number of poles of \( F(z) \) is given by

\[ n = m + 2 = N + 2 \]  

(18a)
and \( F(z) \) will be of the form
\[
F(z) = \frac{(z-z_1)(z-z_2) \ldots (z-z_N) z^2}{(z-p_1)(z-p_2)(z-1)^{N+1}}
\] (18b)

Finally, using Eq. (18b) in Eq. (8) results in an open loop transfer function of the desired form, namely
\[
G(z) = G \frac{(z-z_a)(z-z_o)(z-z_i)(z-z_2) \ldots (z-z_N)}{(z-p_1)(z-p_2)(z-1)^{N+1}}
\] (18c)

The order of \( G(z) \) will be \( N + 3 \), and the degree of its numerator will be one less than that of the denominator.

The lower bound on the allowable gain \( G \) is determined by the proximity of \( z_i \) to the point \( z = 1.0 \). The upper bound on \( G \) is controlled by the location of the poles \( p_1 \) and \( p_2 \), since as \( G \) increases these poles move outside the unit circle. By trial and error, it was found that the poles originating at \( p_1 \) and \( p_2 \) will stay closer inside the unit circle when \( p_1 \) and \( p_2 \) are placed on the real axis between \( z = -1.0 \) and \( z = z_a \). The maximum upper bound on \( G \) is achieved when \( p_1 \) is very close to \( z = -1.0 \) and \( p_2 \) is very close to \( z = z_a \). The root locus of these two poles is shown in Fig. 5.

Figure 6 illustrates the impact of the location of \( z_1 \) on the loop gain margin for the case when \( N = 1 \).

\[ \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{n+1} \\ a_1 & a_0 + a_2 & a_1 + a_3 & \ldots & a_n \\ \vdots & & \ddots & \ddots & \vdots \\ a_n & 0 & 0 & \ldots & a_0 \\ \end{bmatrix} \]

(22)

Let \( \Omega \) be the matrix formed from \( \Omega \) by replacing the first column of Eq. (22) with

\[ \begin{bmatrix} \sum_{i=0}^{n+1} b_i^2 \\ 2 \sum_{i=0}^{n+1} b_i h_i \beta_{i+1} \\ 2 \sum_{i=0}^{n+1} b_i h_i \beta_{i+2} \\ \vdots \\ 2 b_0 b_{n+1} \end{bmatrix} \]

(23)

According to Ref. 2, the integral \( I_n \) is equivalent to the ratio of two determinants as follows:
\[
I_n = \frac{|\Omega_1|}{a_0 |\Omega|}
\] (24)

Using Eqs. (22), (23) and (24) in Eq. (21), the bandwidth vs \( G \) was calculated for the filters and gain values given in Table 1, and plotted in Figs. 7-11. For loops which are similar in their pole-zero location, increasing the number of integrators in \( F(z) \) increases \( B_L \). Adding more poles and/or zeros near \( z = 0 \) has an insignificant impact on the bandwidth. The impact of the location of \( z_1 \) on the bandwidth when \( N = 1 \) is shown in Fig. 6. \( B_L \) was computed for \( G = G_{\text{min}} + 4 \text{ dB} \).
C. Steady-State Error

The input to our z-domain loop (see Fig. 2) can be expressed as

\[
\frac{\theta(s)}{s} = \frac{\theta_0 Tz}{(z-1)^2} + \frac{\theta_2 T^2 z(z+1)}{2(z-1)^3} + \frac{\theta_3 T^3 z(z^2+4z+1)}{3(z-1)^4} + \frac{\theta_4 T^4 z(z^3+11z^2+11z+1)}{4(z-1)^5} + \ldots
\]  

(25)

where \(\theta_0, \theta_2, \theta_3, \theta_4\) represent phase step, ramp, acceleration, and jerk, respectively. Higher orders terms are assumed to be negligible.

The transfer function \(H_e(z)\) is defined as

\[
H_e(z) = \frac{\theta(s)}{z} = \frac{G_Q (z-1)}{z(1+G(z))}
\]  

(26)

where \(G(z)\) is given by Eq. (8) which results in the steady state error

\[
e_{ss} = \lim_{z \to 1} \left(1 - \frac{1}{z}\right) H_e(z) \left(\frac{\theta(s)}{s}\right)^*
\]  

(27)

Let \(F(z)\) be of the form in Eq. (18b). Then, inserting Eqs. (25) and (26) in Eq. (27) and simplifying, we obtain

\[
e_{ss} = \lim_{z \to 1} \left[\frac{\theta_0 Tz}{(z-1)^2} + \frac{\theta_2 T^2 z(z+1)}{2(z-1)^3} + \frac{\theta_3 T^3 z(z^2+4z+1)}{3(z-1)^4} + \frac{\theta_4 T^4 z(z^3+11z^2+11z+1)}{4(z-1)^5} + \ldots \right.
\]

\[
+ \frac{G_Q (z-p_1)(z-p_2)(z-1)^{N+3}}{z^2(z-1)^{N+1}(z-p_1)(z-p_2)} + \ldots
\]

\[+ G(z-x_2)(z-x_3)(z-x_4) \ldots (z-x_N)]
\]

(28)

With \(N = 1, \theta_3\) (phase acceleration) will be tracked with finite steady-state error, but \(\theta_4\) (phase jerk) will generate infinite steady-state error. With \(N = 2, \theta_4\) will be tracked with finite steady-state error. With \(N \geq 3\) the steady-state error due to phase jerk will be zero. Table 2 lists values of steady-state error for different values of \(N\). Note that the proximity of \(z_1\) to the point \(z = 1.0\) increases the value of the steady-state error.

Steady-state phase error, \(\phi_{ss}\), can be readily obtained from \(e_{ss}\), using the relation

\[
\phi_{ss} = \frac{e_{ss}}{G_Q T}
\]  

(29)

which is equivalent to

\[
\phi_{ss} = \frac{e_{ss} T(1-g)^2}{2G}
\]  

(30)

by using Eq. (9).

D. Transient Response

In general, for a digital loop with a large number of poles and zeros, it is not easy to calculate the transient response. If, for some specific gain value, the loop has only two dominant poles, then the natural frequency, damping ratio and settling time can be assessed from the second-order loop equations. Otherwise, an inverse z-transform operation has to be performed on the product of the loop transfer function and the step input. There are three methods available for this inverse z-transformation: integration, partial fraction expansion and long division. In our analysis, we have chosen the last of the three.

By performing synthetic (long) division on the ratio of two polynomials in

\[
e(z) = \frac{G_Q (z-1)}{z(1+G(z))} \times \frac{\theta_0 Tz}{(z-1)^2}
\]

the time response of the loop with a step input is obtained at time \(t_1, t_2, t_3, \ldots\). The settling time \(T_s\) is determined when the magnitudes of \(e_i\) become less than, say, 5% of the initial value for all \(t_f, t_f+1, \ldots\). Plots of \(T_s\) vs \(G\) for a loop with filters specified in Table 1 are shown in Figs. 7–11. As can be seen from these figures, the settling time increases very sharply when the two poles on the left portion of the real axis migrate close to the unit circle (see Fig. 5). The impact of the location of \(z_1\) on \(T_s\) for \(N = 1\) is shown in Fig. 6.

IV. Conclusion

In this article, a model for a general digital-phase locked loop is proposed. The impact of a digital filter on the stability, gain margin, noise-equivalent bandwidth and time response is investigated. A specific type of digital filter is proposed which, for any desired number of integrators, has a minimum number of zeros. The strategy adopted in this analysis in optimizing
the loop performance with the digital filter consists of two steps. First, the lower and upper bounds of the allowable loop gain are obtained by placing the filter's zeros close to the point \( z = 1.0 \) and the two filter's poles close to \( z = -1.0 \) and \( z = z_a \). Then, the desired bandwidth and settling time are obtained by selecting the corresponding value for the loop gain.

Table 2 lists steady-state error values vs the number of integrators of the loop filter. Making the number of integrators equal to 3, the loop can track phase jerk with zero steady-state error (at loop update instants). Figures 7 through 11 show the variation of the noise bandwidth and settling time with changing gain for different loop types.

References


Table 1. Proposed placement of poles and zeros for loop filters

<table>
<thead>
<tr>
<th>Location of Poles and Zeros(^1) of (F(z))</th>
<th>Number of Integrators of (F(z))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_1)</td>
<td>0.960 0.960 0.960 0.970</td>
</tr>
<tr>
<td>(z_2)</td>
<td>0.960 0.930 0.960 0.960</td>
</tr>
<tr>
<td>(z_3)</td>
<td>0.930 0.940</td>
</tr>
<tr>
<td>(z_4)</td>
<td>0.940</td>
</tr>
<tr>
<td>(p_1)</td>
<td>-0.173 -0.173 -0.173 -0.173</td>
</tr>
<tr>
<td>(p_2)</td>
<td>-0.999 -0.999 -0.999 -0.999</td>
</tr>
<tr>
<td>(p_3)</td>
<td>1.000 1.000 1.000 1.000</td>
</tr>
<tr>
<td>(p_4)</td>
<td>1.000 1.000 1.000 1.000</td>
</tr>
<tr>
<td>(p_5)</td>
<td>1.000 1.000 1.000</td>
</tr>
<tr>
<td>(p_6)</td>
<td>1.000</td>
</tr>
<tr>
<td>Approximate range of (G)</td>
<td>0.001 0.001 0.01 0.04 0.04</td>
</tr>
</tbody>
</table>

\(^1\) \(F(z)\) contains also \(n - m\) zeroes at \(z = 0\). In our analysis \(n - m = 2\).

Table 2. Steady-state error signal vs loop type

<table>
<thead>
<tr>
<th>Loop Type</th>
<th>Steady State Error, (e_{sy})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Phase Ramp</td>
</tr>
<tr>
<td>1</td>
<td>(\theta_d (1 - p_1)(1 - p_2))</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
Fig. 1. Closed loop hybrid s/z diagram

Fig. 2. Equivalent closed loop z-domain diagram
Fig. 3. Direct form I realization of $F(z)$

Fig. 4. Direct form II realization of $F(z)$
Fig. 5. Root-locus in the complex z-plane. $N =$ number of integrators of $F(z)$. 
Fig. 6. $G_{\text{max}}$, $G_{\text{min}}, B_L$, and $T_s$ vs location of $z_1$, for $F(z)$ with 1 integrator

Fig. 7. Bandwidth and settling time vs loop gain. Number of Integrators: 0.

Fig. 8. Bandwidth and settling time vs loop gain. Number of Integrators: 1.
Fig. 9. Bandwidth and settling time vs loop gain. Number of integrators: 2.

Fig. 10. Bandwidth and settling time vs loop gain. Number of integrators: 3.

Fig. 11. Bandwidth and settling time vs loop gain. Number of Integrators: 4.