A Note on the Wideband Gaussian Broadcast Channel

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It is well known that for the Gaussian broadcast channel, timeshared coding is not as efficient as more sophisticated "broadcast" coding strategies. However, in this article we will show that the relative advantage of broadcast coding over timeshared coding is small if the signal-to-noise ratios of both receivers are small. One surprising consequence of this is that for the wideband Gaussian broadcast channel, which we shall define, broadcast coding offers no advantage over timeshared coding at all, and so timeshared coding is optimal.

I. Introduction

T. M. Cover (Ref. 1) introduced the "broadcast channel" with one transmitter and two (or more) different receivers. Following Ref. 4, we ask the following question about a broadcast channel: certain common information is to be communicated simultaneously to both receivers. How much additional information can be communicated to the better receiver at the same time?

For channels like Gaussian channels, where one receiver is just a degraded version of the other, one obvious approach is timeshared coding: devote a fixed fraction of the total transmission time to sending the common information, coded for the weaker channel. This information will be comprehensible to both receivers. During the remaining time, transmit additional information coded for the stronger receiver. This information will not be comprehensible to the weaker receiver.

But in Ref. 1, Cover introduced a technique called broadcast coding, and showed that, in general, broadcast coding achieves greater, often much greater, data rates than time sharing. Later El Gamal and Cover (Ref. 2) found that broadcast coding cannot be further improved upon.

In this article we will discuss the Gaussian broadcast channel. For this channel, we will show that the relative advantage of broadcast coding over timeshared coding is small if the signal-to-noise ratios of both receivers are small. One surprising consequence of this is that for the wideband Gaussian broadcast channel, which we shall define, broadcast coding offers no
advantage over timeshared coding at all, and so timeshared
coding is optimal.

II. The Gaussian Channel: A Review

A Gaussian channel is a discrete-time memoryless channel
model whose input $X$ and output $Y$ are related by $Y = X + Z$,
where $Z$ is a mean zero Gaussian random variable independent
from $X$. If the input is constrained by $E(X^2) \leq S$, and if the
variance of $Z$ is denoted by $\sigma^2$, it is well known that the chan-
nel capacity depends only on the ratio $x = S/\sigma^2$, which is
called the signal-to-noise ratio, and is given by the formula

$$C(x) = \frac{1}{2} \log(1 + x)$$  \hspace{1cm} (1)

In Eq. (1), $C(x)$ represents the maximum possible amount of
information (measured in bits, nats, etc., depending on the
base of the logarithm) which can be reliably transmitted per
channel use; in the usual physical sense, $C(x)$ is dimensionless.

Equation (1) can be used to derive the following formula
for the capacity of a continuous-time, band- and power-
limited Gaussian channel model:

$$C = B \log \left(1 + \frac{P}{N_0 B}\right)$$  \hspace{1cm} (2)

where $B$ is the channel bandwidth in Hertz, $P$ is the average
transmitter power in Watts, and $N_0$ is the noise spectral den-
sity in Watts per Hertz. The transition from Eq. (1) to Eq. (2)
is explained in Ref. 3 (Chapter 4), for example. In Eq. (2), $C$
represents the maximum possible information which can be
reliably transmitted per unit of time; the physical dimensions
of $C$ are $\text{sec}^{-1}$.

If in Eq. (2) we assume natural logarithms and pass to the
limit as the bandwidth $B$ approaches infinity, we obtain

$$C = \frac{P}{N_0}$$  \hspace{1cm} (3)

which is the well-known formula for the capacity of the infinite
bandwidth white Gaussian channel. The units in Eq. (3) are
nats per second.

III. The Gaussian Broadcast Channel

In Ref. 1, Cover introduced a discrete-time memoryless
channel model with one transmitter and two receivers, which
he called a Gaussian broadcast channel. This channel has one
input $X$, and two outputs $Y_1$ and $Y_2$, related by

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

where now $Z_1$ and $Z_2$ are mean zero Gaussian random vari-
ables, and $X$, $Z_1$, and $Z_2$, are independent. Let us denote by
$\sigma_1^2$ and $\sigma_2^2$ the variances of $Z_1$ and $Z_2$, respectively, and assume
that $\sigma_1^2 \leq \sigma_2^2$, so that $Y_1$ is received more reliably than $Y_2$. If
the channel input $X$ is constrained as in Section I by $E(X^2) \leq S$,
then separately channels 1 and 2 have capacity $C(x_1)$ and
$C(x_2)$, respectively, where $x_1 = S/\sigma_1^2$ and $x_2 = S/\sigma_2^2$.

In Ref. 4 a Gaussian broadcast channel was used to model
deep-space communications in the presence of weather uncer-
tainties; the high signal-to-noise ratio corresponds to good
weather, and the low signal-to-noise ratio, to bad weather. The
problem posed there was the following. Suppose the weather
on earth is unknown to a distant spacecraft, and that data
must be sent to earth so that even in bad weather, certain
minimal but critical information will get through; but if the
weather is good, additional information will be received.

Motivated by this point of view, we state the fundamental
question about broadcast channels in the following somewhat
unusual way. Suppose we wish to send certain information,
called the common information, simultaneously to both
receivers. If we do this, how much extra information, called
bonus information, can we send to the better receiver at the
same time?

One approach to this problem is the timesharing approach,
in which the transmitter devotes a fixed fraction $1 - \rho$ (where
$0 < \rho < 1$) of the total transmission time to sending the com-
mon information. During this time the information is coded
for the weaker receiver. This information will also be compre-
hensible to the stronger receiver. By Eq. (1), during this com-
mon time, information can be transmitted at a maximum rate
of $C(x_1)$. During the remaining fraction $\rho$ of the transmis-
sion time, the transmitter sends bonus information to the stronger
channel, at the rate $C(x_2)$. This will not be comprehensible to
the weaker receiver.

It follows that for the timesharing strategy, the data rates
will be

$$\text{Common Rate} = (1 - \rho) C(x_2)$$
$$\text{Bonus Rate} = \rho C(x_1)$$  \hspace{1cm} (4)

and the parameter $\rho$ can be selected arbitrarily by the trans-
mitter.

Cover showed, however, that it is possible to do better
than timesharing. Using a technique called broadcast coding,
he showed that for any choice of the parameter \(\alpha, 0 < \alpha < 1\), the following rates are achievable:

\[
\text{Common Rate} = C(x_2) - C(\alpha x_2) \\
\text{Bonus Rate} = C(\alpha x_1)
\]

(Actually Cover gave the common rate in the form

\[
C\left[\frac{(1 - \alpha)x_2}{(1 + \alpha x_2)}\right],
\]

but it is an easy exercise to show that this is the same as we have given in Eq. (5).) Later El Gamal and Cover (Ref. 2) showed that in fact no improvement over Eq. (5) is possible, so that the region of the first quadrant bounded by the curve given parametrically by Eq. (5) is now called the capacity region of the Gaussian Broadcast Channel (see Fig. 1).

Motivated by the discussion in Section 1, let us pass from the discrete-time Gaussian broadcast channel to the continuous-time band- and power-limited Gaussian broadcast channel. The resulting expressions are for timesharing:

\[
\text{Common Rate} = (1 - \rho) B \log \left(1 + \frac{P}{N_2 B}\right) \\
\text{Bonus Rate} = \rho B \log \left(1 + \frac{P}{N_1 B}\right)
\]

(6)

and for broadcast coding:

\[
\text{Common Rate} = B \log \left(1 + \frac{P}{N_2 B}\right) - B \log \left(1 + \alpha \frac{P}{N_2 B}\right) \\
\text{Bonus Rate} = B \log \left(1 + \alpha \frac{P}{N_1 B}\right)
\]

(7)

where \(P\) is the transmitter power, \(B\) is the transmission bandwidth, and \(N_1, N_2\), are the noise spectral densities for the two receivers. In Eqs. (6) and (7) the units are nats per second.

To investigate wideband Gaussian broadcast channels, we pass as before to the limit as \(B \to \infty\). The results follow easily from Eqs. (6) and (7) for wideband timesharing:

\[
\text{Common Rate} = (1 - \rho) \frac{P}{N_2} \\
\text{Bonus Rate} = \rho \frac{P}{N_1}
\]

and for wideband broadcast coding:

\[
\text{Common Rate} = (1 - \alpha) \frac{P}{N_2} \\
\text{Bonus Rate} = \alpha \frac{P}{N_1}
\]

(9)

We thus reach the surprising conclusion that for wideband Gaussian broadcast channels, broadcast coding offers no advantage over timesharing. (Actually, this was mentioned but not further investigated in Ref. 4.) We investigate this interesting phenomenon more closely in the next section.

IV. A More Detailed Analysis

In this section we will see that the reason wideband broadcast coding offers no advantage over wideband timesharing is that, for a given common rate, the bonus rates in Eqs. (4) and (5) are nearly equal, when the "good" SNR \(x_1\) is small. More precisely, we have the following:

\textbf{Theorem:} If \(\alpha\) and \(\rho\) are chosen so that the common rates in Eqs. (4) and (5) are equal, then

\[
\frac{\text{Broadcast bonus rate (BBR)}}{\text{Timesharing bonus rate (TBR)}} = \frac{C(\alpha x_1)}{\rho C(x_1)}
\]

\[
\leq \frac{x_1}{C(x_1)} \cdot \frac{C(x_2)}{x_2} = \frac{x_1}{x_2} \cdot \log \left(1 + \frac{x_2}{x_1}\right)
\]

\textbf{Corollary 1:} Since \(\log(1 + x_2) \ll x_2\), we also have

\[
\frac{\text{BBR}}{\text{TBR}} \leq \frac{x_1}{\log(1 + x_1)}
\]

independent of \(x_2\). Thus also

\[
\lim_{x_1 \to 0} \frac{\text{BBR}}{\text{TBR}} = 1
\]

again independent of \(x_2\).
Corollary 2: For the continuous time channel, the corresponding result is

\[
\frac{\text{BBR}}{\text{TBR}} \leq \frac{N_2}{N_1} \cdot \frac{\log(1 + P/N_2 B)}{\log(1 + P/N_1 B)}
\]

\[
< \frac{P/N_1 B}{\log(1 + P/N_1 B)}
\]

\[
\rightarrow 1 \quad \text{as} \quad B \rightarrow \infty
\]

Proof of Theorem: For the two common rates to be equal, we have, from Eqs. (4) and (5), that

\[
C(ax_2) = \rho C(x_2)
\]

(10)

On the other hand, the ratio of the bonus rates is

\[
\frac{C(ax_1)}{\rho C(x_1)}
\]

(11)

Combining Eqs. (10) and (11), we see that, for a fixed common rate, the ratio of the bonus rates is

\[
\frac{C(ax_1)}{C(ax_2)} \cdot \frac{C(x_2)}{C(x_1)}
\]

The desired result now follows from the fact that the function \( C(ax_1)/C(ax_2) \) is a decreasing function of \( x_1/x_2 \) as \( x_1/x_2 \) approaches 0.

We conclude with a brief discussion of the shape of the broadcast capacity regions as a function of \( x_1 \) and \( x_2 \). It is useful to normalize the parametric curves described by Eq. (5) by dividing the common rate by its maximum value \( C(x_2) \), and the bonus rate by its maximum value \( C(x_1) \):

\[
\text{Normalized Common Rate (NCR)} = 1 - \frac{C(ax_2)}{C(x_2)}
\]

(12)

\[
\text{Normalized Bonus Rate (NBR)} = \frac{C(ax_1)}{C(x_1)}
\]

For a given value of \( x_1 \), the parametric curves described by Eq. (12) vary monotonically outward from \( x_2 = x_1 \), in which case they reduce to

\[
\text{NCR} = 1 - \frac{C(ax_2)}{C(x_2)}
\]

\[
\text{NBR} = \frac{C(ax_1)}{C(x_1)}
\]

which is just the timesharing straight line, to \( x_2 = 0^+ \), in which case they reduce to

\[
\text{NCR} = 1 - \alpha
\]

\[
\text{NBR} = \frac{C(ax_1)}{C(x_1)}
\]

Thus for a given good SNR \( x_1 \), broadcast coding offers the largest relative advantage over timesharing when \( x_2 \) is small, the smallest relative advantage (none at all) when \( x_2 = x_1 \). Of course, as we have seen, when \( x_1 \) is small, even the largest relative advantage is quite small. In Fig. 2, we have graphed the outer \( x_2 = 0 \) and inner \( x_2 = x_1 \) envelopes for several values of \( x_1 \).

References


Fig. 1. The capacity region of some Gaussian broadcast channels.
Fig. 2. The extreme capacity regions ($x_2 = 0$ and $x_2 = x_1$) for several values of $x_1$. 