Sequential Ranging With the Viterbi Algorithm

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The performance of the sequential ranging system can be improved by using a maximum likelihood receiver; however, the complexity grows exponentially with the number of components \(N\) needed to determine the range unambiguously. A new truncated maximum-likelihood receiver, based on the Viterbi decoder for convolutional codes, is presented and is shown to achieve a maximum-likelihood performance while having a fixed complexity independent of \(N\). The improvement in signal-to-noise ratio, compared to the present receiver, is 1.5 dB for \(P_e < 10^{-2}\).

1. Introduction

Ranging systems for deep space applications achieve the required resolution by transmitting, either simultaneously or sequentially, a multi-component signal. Goldstein (Ref. 1) described a sequential ranging system, which transmits \(N + 1\) squarewaves of increasing periods \(T, 2T, \ldots, 2^N T\). The highest frequency component yields the most accurate measurement, but with an added distance of \(M \cdot R\), where \(R\) is proportional to \(T\) and \(M\) is an unknown integer. The other \(N\) components are used to estimate \(M\) and thus remove the ambiguity.

In the above system, the estimation is done sequentially, that is, at each step a binary number \(a_k\) is estimated from the signal component which is present at that step. The sequence \(a_1, \ldots, a_N\), considered as a binary representation of \(M\), yields an estimate of \(M\).

The performance of the system—that is, the probability of estimating \(M\) correctly—can be improved by using a maximum-likelihood estimator, which estimates the whole sequence \(a_1, \ldots, a_N\) simultaneously. However, the complexity of such a system is proportional to \(2^N\) and is not practical for large \(N\).

We will present a suboptimum estimation procedure, which outperforms the sequential receiver and approaches asymptotically, as the signal-to-noise ratio increases, the performance of the maximum-likelihood receiver. This method is based on the Viterbi algorithm for decoding convolutional codes of short constraint length \((\nu)\), and has a complexity of the order of \(2^\nu\), no matter how large \(N\) is. The improvement in the signal-to-noise ratio required to achieve a given error probability \(P_e\) is 1.5 dB throughout the range of interest \((P_e < 10^{-2})\).
II. The Sequential Receiver

The time-of-flight (TOF) of the signal at $t_0$ can be represented by

$$\text{TOF} = (M + \epsilon) T$$

(1)

where $T$ is the period of the first (highest frequency) squarewave, $M$ is a positive integer and $0 \leq \epsilon < 1$.

Let $\{a_N a_{N-1}, \cdots, a_1\}$ be the binary representation of $M$, that is

$$M = \sum_{k=1}^{N} a_k 2^{(k-1)}$$

(2)

where $a_k$ is 0 or 1.

To measure the TOF, and therefore the range, it is enough to measure $\epsilon$ and $\{a_1, \cdots, a_N\}$. The present receiver does this sequentially.

We start by transmitting the $T$-period squarewave to obtain an estimate $\hat{\epsilon}$ of $\epsilon$. The receiver correlates the incoming signal with the receiver coder squarewave and its 90 deg-shift to obtain a pair of outputs $x_k, y_k$, from which $\epsilon$ is estimated. The correlator is then shifted by $\hat{\epsilon}$ to have a phase of $\hat{\epsilon} - \epsilon$. We will assume that the integration time is long enough to obtain $\epsilon - \hat{\epsilon} = 0$. To estimate $a_1, \cdots, a_N$, squarewaves of periods $2^{k} T$, $k = 1, \cdots, N$ are transmitted sequentially. The outputs of the in-phase and quadrature correlators at the $k$th step are

$$x_k = s_k + n_k$$
$$y_k = r_k + m_k$$

(3)

respectively, where $n_k$ and $m_k$ are independent white gaussian noise samples of zero mean and variance $\sigma^2$.

$$s_k = \begin{cases} 1 - a_k, & 0 \leq a_k < 2 \\ a_k - 3, & 2 \leq a_k < 4 \end{cases}$$

$$r_k = \begin{cases} a_k, & 0 \leq a_k < 1 \\ 2 - a_k, & 1 \leq a_k < 3 \\ a_k - 4, & 3 \leq a_k < 4 \end{cases}$$

(4)

and $a_k$ depends on $a_1, \cdots, a_{k-1}$.

It was shown that the probability of error is minimized by shifting the correlator waveform by 90 deg at the $k$th step whenever $\hat{a}_{k-1} = 1$, where $\hat{a}_{k-1}$ is the estimate of the previous step. Thus

$$a_k = 2a_k + \sum_{j=1}^{k-1} (a_j - \hat{a}_j) 2^{(j+1-k)}$$

(5)

and we estimate

$$\hat{a}_k = \begin{cases} 0 & \text{if } x_k \geq 0 \\ 1 & \text{if } x_k < 0 \end{cases}$$

(6)

The procedure is terminated when $\hat{a}_N$ is obtained. The resulting

$$\hat{M} = \sum_{k=1}^{N} \hat{a}_k 2^{k-1}$$

(7)

and $\hat{\epsilon}$ yield the measured TOF.

III. Maximum-Likelihood Estimation

The "estimate-and-shift" sequential method has a strong error propagation property. Suppose the estimate $\hat{a}_i$ of $a_i$ is in error, e.g., $a_i = 0$ but $\hat{a}_i = 1$. The correlator waveform is shifted by 90 deg at the second step, therefore the no-noise outputs will be $A'$ if $a_2 = 0$ and $B'$ if $a_2 = 1$ instead of $A$ and $B$, respectively (Fig. 1). Since the receiver estimates $a_2 = 0$ whenever $x_2 \geq 0$, the probability of wrongly estimating $a_2$ is $\frac{1}{2}$. It can be shown that the error in estimating $a_i$ will adversely affect the estimation

\[ \text{Fig. 1. Correlators output at second step: A or B if } \hat{a}_i = a_i \text{ and } A' \text{ or } B' \text{ if } \hat{a}_i \neq a_i. \]
of \(a_0, a_1, \ldots\). However, the effect will diminish as \(k\) increases.

This error propagation does not affect the performance of the system, since one wrong estimation is enough to cause an error in the measurement of the TOF. However, since \(a_k\) affects all \((x_n, y_n)\) for \(n \geq k\), an estimate of \(a_k\) based on \((x_n, y_n; n \geq k)\) is superior to one which is performed at the \(k\)th step, and therefore depends on \((x_k, y_k)\) only.

Assume, as before, that \(\widehat{\epsilon} \approx \epsilon\) and let

\[
x = (x_1, \ldots, x_N) \\
y = (y_1, \ldots, y_N) \\
a = (a_1, \ldots, a_N)
\]

be the correlators output vectors and the TOF binary vector respectively. By Bayes rule we have

\[
p(a|x, y) = \frac{p(x, y|a)}{p(x, y)} p(a) \tag{8}
\]

and the maximum-likelihood estimate is the binary vector \(a^*\) which maximizes

\[
p(x, y|a) = \frac{1}{(2\pi \sigma^2)^{N/2}} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{k=1}^{N} [(x_k - s_k)^2 + (y_k - r_k)^2] \right\} \tag{9}
\]

or equivalently minimizes

\[
\ell(a) = \sum_{k=1}^{N} [(x_k - s_k)^2 + (y_k - r_k)^2] \tag{10}
\]

where \(s_k, r_k, k = 1, \ldots, N\) can be expressed in terms of \(a\) (Eqs. 4 and 5).

Note that the \(\widehat{\alpha}_j, j = 1, \ldots, k - 1\) that appear in \(a_k\) (and hence in \(s_k\) and \(r_k\)) are on-the-spot estimates which determine whether the correlator waveforms are shifted in the next step. They depend on the \(x_i/\)'s alone and not on \(a\), and are known to the receiver when \(\ell(a)\) is evaluated.

Since we have to choose the most likely of \(2^N\) possibilities, the complexity grows as \(2^N\), and the method is not practical for large \(N\).

**IV. Truncated Maximum-Likelihood Estimate and the Viterbi Algorithm**

The contribution of \(a_k\) to \((s_n, r_n)\) is halved at every successive step, since its coefficient in \(\alpha_n\) is proportional to \(2^{(k-n)}\) (Eq. 5). In other words, the value of \(a_k\) affects the correlator outputs for all \(n \geq k\); however, this effect diminishes exponentially as \(n\) increases. Thus, there exists some integer \(v\), which depends on the signal-to-noise ratio, such that the contribution of \(a_k\) to \((s_n, r_n)\) for \(n > k + v\) is negligible. We therefore can approximate Eq. (5) by

\[
a_k = 2a_k + \sum_{j=k-v}^{k-1} (a_j - \widehat{a}_j) 2^{(i+1-k)} \tag{11}
\]

for all \(k > v + 1\).

Thus each \((s_k, r_k)\) depends on \(a_k\) and the previous \(v\) components and we have a finite state machine with \(2^v\) states, corresponding to all possible binary vectors \((a_{k-1}, \ldots, a_{k-v})\), and two outputs per state depending on \(a_k\). The progress of this machine, during few successive steps can be depicted by its trellis diagram (Fig. 2).

**Fig. 2. Trellis diagram for \(v = 3\) during steps \(k\) and \(k + 1\)**
During step $k$, each state $(a_{k-1}, \cdots, a_{k+n})$ can advance to one of two states $(0, a_{k-1}, \cdots, a_{k+n+1})$ or $(1, a_{k-1}, \cdots, a_{k+n+1})$ depending whether $a_k$ is 0 or 1. The corresponding correlation outputs $(s_k, r_k)$ will depend on the starting state as well as on $a_k$.

The similarity to convolutional coding with a constraint length of $\nu$ is immediate. Many decoding procedures have been developed for convolutional codes, however, for short constraint length ($\nu < 10$) the Viterbi algorithm (Ref. 2) is the most efficient, and is actually a maximum-likelihood estimate of the truncated estimation problem (Ref. 3).

The Viterbi algorithm can be briefly described as follows: With every state we associate a metric (accumulated likelihood function up to this step) and a survivor (the most likely sequence leading to this state). At step $k$, each state $S_{k-1} = (a_{k-1}, \cdots, a_{k+n-1})$ can be reached from two states $S_{k-1}^\delta = (a_{k-1}, \cdots, a_{k+n-1}, \delta)$, where $\delta$ is 0 or 1. After the received signal is correlated to yield $(x_k, y_k)$, we compare the two possible ways to reach $S_k$ and keep the most likely of them, by properly updating the metric and the survivor of $S_k$. This is done for each one of the $2^n$ states, and is repeated every step. The final decision is made after the $N$th (last) step, by comparing the $2^n$ metrics and selecting the survivor of the largest, to yield the most likely estimate of $a$.

Thus a ranging receiver based on the Viterbi algorithm can yield a (truncated) maximum-likelihood performance with a complexity of $2^n$, which is independent of the number of components $N$ to be estimated.

**V. Error Probabilities**

The signal is received in the presence of additive white gaussian noise of zero mean and spectral density $N_0$. If the signal power is $S$ and the integration time of each component is $\tau_k = \tau, k = 1, 2, \cdots, N$ the probability that the sequential receiver will correctly estimate the whole sequence $(a_1, \cdots, a_N)$, is given by

$$P_e = 1 - \left[1 - \text{erfc} \left(\frac{S\tau}{2N_0}\right)^{1/2}\right]^\nu$$

where

$$\text{erfc}(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \exp(-x^2/2) \, dx$$

(12)

If $\Phi - \epsilon \neq 0$ the performance is degraded; however, we can assume that the integration time of the highest frequency component is long enough to obtain $\Phi \approx \epsilon$. An analytic expression for the performance of the maximum-likelihood receiver or for the Viterbi algorithm cannot be obtained in closed form. Therefore, the error probabilities for various signal-to-noise ratios were obtained by computer simulations. The noise was generated by a multiplicative congruential generator (Ref. 4), and quantized in steps of $\sigma^2/32$, where $\sigma^2$ is the noise-to-signal ratio.

The results for the maximum-likelihood estimate of 10 components ($N = 10$), and a truncation to $\nu = 5$ are shown in Fig. 3 together with the performance of the sequential receiver. The improvement gained by maximum-likelihood estimation compared to sequential estimation is 1.5 dB for the measured range of error probabilities. This improvement is also achieved by the Viterbi algorithm (with $\nu = 5$ truncated memory and therefore a
smaller complexity), for signal-to-noise ratios which yield 
P_s < 10^{-4}.

V. Conclusions

A truncated maximum likelihood receiver for sequential ranging has been presented. The performance of the sequential ranging system can be improved by using maximum-likelihood techniques; however, the complexity grows exponentially with the number of components N needed to determine the range unambiguously. The suggested method, which is based on the Viterbi decoder for convolutional codes, performs like the maximum-likelihood receiver while having a finite complexity independent of N. The improvement in signal-to-noise ratio, compared to the present receiver, is 1.5 dB.

References


