Combinational Complexity Measures as a Function of Fan-Out

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If $C_s(f_1, \ldots, f_L)$ is the fan-out $s$ combinational complexity of the functions $f_1, f_2, \ldots, f_L$ with respect to straight-line algorithms (or combinational machines) of fan-in $r$, then it is shown that

$$C_s(f_1, \ldots, f_L) \leq C_s(f_1, \ldots, f_L)$$

$$\leq \left(\frac{d(r-1)}{s-1} + 1\right) C_s(f_1, \ldots, f_L) + \frac{d}{s-1} (L - N)$$

where $N$ is the number of variables on which $f_1, \ldots, f_L$ depend and $d = C_s(I)$ where $I$ is the identity function in one variable. Thus, a well-designed combinational machine or algorithm will not have a fan-out which is more than several times its fan-in.

I. Introduction

In this paper we develop bounds on the fan-out $s$ combinational complexity of functions. These bounds show that the combinational complexity of functions has a weak dependence on fan-out when $s \gg 2$.

II. Bounds on Combinational Complexity

Before we develop the promised bounds, we state the following definitions which are needed in the sequel.

**Definition 1.** Let $\Omega$ be a set of functions over the set $\Sigma$, such that if $h_1 : \epsilon \Omega$, then $h_1 : \Sigma^n \rightarrow \Sigma$. Let

$$\Gamma = \Sigma \cup \{X_1, X_2, \cdots, X_k\}$$

Then, an $(\Omega, \Gamma)$ algorithm (or "straight-line" algorithm) is a $K$-tuple $\beta = (\beta_1, \beta_2, \cdots, \beta_K)$ where either $\beta_k \in \Gamma$ or $\beta_k = (h_1; k_1, k_2, \cdots, k_n)$, $h_1 \in \Omega$, $1 \leq k_1 < k$. The set of functions $(\beta_1, \beta_2, \cdots, \beta_K)$ is associated with $\beta$ where $\beta_k = \beta_k$ if $\beta_k \in \Gamma$ and $\beta_k = h_i (\beta_{k_1}, \cdots, \beta_{k_{n_1}})$ if

$$\beta_k = (h_i; k_1, k_2, \cdots, k_n)$$
An algorithm $\beta$ is said to compute the functions
\[ f_t : \Sigma^m \rightarrow \Sigma, \quad m_i \leq N, \quad 1 \leq t \leq L. \]
if there exist $\beta_{m_1}, \ldots, \beta_{m_L}$ such that $f_t = \beta_{m_t}$.

The fan-in of $\Omega$ is
\[ r = \max_i n_i \]
where $h_i : \Omega, h_i : \Sigma^m \rightarrow \Sigma$. If $\beta$ computes $f_1, f_2, \ldots, f_L$ where $f_t = \beta_{m_t}, 1 \leq t \leq L$, let $\gamma_i$ the number of steps of $\beta$ which use $\beta_i$, if $\beta_i \notin \Sigma$, and $\gamma_i = 0$, otherwise and let $\theta_i = \gamma_i$, $i \neq m_i, m_{i_2}, \ldots, m_L$ and $\theta_i = \gamma_i + 1$ otherwise. Then, the fan-out of $\beta$ is
\[ s = \max_i \theta_i \]

**Definition 2.** The combinational complexity with fan-out $s$ of
\[ f_t : \Sigma^m \rightarrow \Sigma, \quad 1 \leq t \leq L, \quad C_s(f_1, \ldots, f_L) \]
is the smallest number of steps $\beta_1 \land \Gamma$ of any $\Omega(\Gamma)$ algorithm which computes these functions, if one such exists; otherwise $C_s(f_1, \ldots, f_L)$ is $\infty$. Associated with any $\Omega(\Gamma)$ algorithm is a graph $G$ in which vertices correspond to steps of the algorithm and edges are directed and ordered from vertices corresponding to $\beta_{k_1}, \ldots, \beta_{k_{n_i}}$, to the vertex corresponding to $\beta_k$ if $\beta_k = (h_i, \beta_{k_1}, \ldots, \beta_{k_{n_i}})$. Vertices corresponding to steps $\beta_k \in \Gamma$ are called source vertices.

Combinational machines are circuits which correspond to the graphs of $\Omega(\Gamma)$ algorithms in which $\Sigma = \{0, 1\}$ and $\Omega$ is a set of Boolean functions; thus, there is an equivalence between combinational machines and straight-line algorithms. These algorithms are called "straight-line" because they do not permit loops or conditional branching. We now state the principal result of this article.

**Theorem.** Let $f_1, \ldots, f_L$ be distinct functions over $\Sigma$ which depend on $N$ variables. Let $\Omega$ have fan-in $r$ and let it be such that an $\Omega(\Gamma)$ algorithm exists for the identity function $I$ in one variable. Then
\[ C_s(f_1, \ldots, f_L) \leq C_s(f_1, \ldots, f_L) \]
\[ \leq \frac{d(r - 1)}{s - 1} + 1 \]
\[ \times C_s(f_1, \ldots, f_L) + \frac{d}{s - 1}(L - N) \]
where $d = C_s(I)$.

**Proof.** Let $\beta$ be a straight-line algorithm with fan-out $s$ which computes $f_1, \ldots, f_L$ with $C_s(f_1, \ldots, f_L)$ operations. The directed graph of $\beta$ has $N$ source vertices and $L$ vertices identified with the distinct functions $f_1, \ldots, f_L$. To the graph $G$ of $\beta$ add $L$ vertices with edges directed into them from the vertices identified with these functions. The number of edges incident upon vertices in this new graph $G'$ is at most $rC_s(f_1, \ldots, f_L) + L$ since each of the original vertices has at most $r$ edges directed into them. Thus, if $\theta_i$ edges are directed away from the $i$th vertex of $G'$ then
\[ \sum_i \theta_i \leq rC_s(f_1, \ldots, f_L) + L \]
where the sum is over all vertices except those associated with constants.

Since $\Omega$ is complete, the identity function on one variable $I(x)$ can be constructed with some number, e.g., $d$, of elements from it with fan-out $s$. For each $i$, if the $i$th vertex of the graph $G'$ has $\theta_i$ edges directed away from it, we can add $h(\theta_i, s)$ copies of the algorithm realizing $I(x)$ to produce a graph $G''$ which has fan-out $s$. Here
\[ h(\theta_i, s) \leq \frac{\theta_i - 1}{s - 1} \]
so the number of elements in $G''$ is bounded above by
\[ \frac{d}{s - 1} \sum_i (\theta_i - 1) + C_s \]
where the sum on $i$ is taken over all vertices of $G$ including all source vertices other than those associated with constants. Since $C_s(f_1, \ldots, f_L)$ is the minimum number of operations required to realize $f_1, \ldots, f_L$, with fan-out $s$, it follows that
\[ C_s(f_1, \ldots, f_L) \leq \frac{d}{s - 1} (rC_s(f_1, \ldots, f_L)) + L - C_s(N) + C_s \]

The left-hand equality of the theorem follows since $C_s(f_1, \ldots, f_L)$ is a non-increasing function of $s$. QED.

The significance of this result is that all of the complexity measures $C_2, C_3, \ldots, C_s$, differ by at most a constant. Also, $C_s$ approaches $C_s$ with increasing $s$ when $r$ is fixed. For many sets $\Omega, d = 1$; for example, this is true for the set of addition, subtraction, multiplication and divi-
sion over the reals and the set of AND, OR and NOT over the set \( \{0, 1\} \). However, \( d = 2 \) for the set \( \Omega \) containing only NAND over \( \{0, 1\} \).

The combinational complexity of a function with fan-out 1, \( C_1 \), can differ substantially from its combinational complexity with unlimited fan-out. Subbotovskaya (Ref. 1) has shown that the Boolean function \( f(x_1, \cdots, x_N) = X_1 \oplus \cdots \oplus X_N \) where \( \oplus \) denotes the EXCLUSIVE OR has \( C_1(f) > a_1N^{3/2} \) for some constant \( a_1 \) when \( \Omega \) consists of AND, OR and NOT and \( C_\infty(f) < a_2N \) for some other constant \( a_2 \).

Reference